# Local Cohomology

## Vanishing Theorems and Minimal Generation of Ideals

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**Abstract**

This paper gives a short expository overview of local cohomology with an emphasis on vanishing theorems. A particular application to minimal generation of ideals up to radical is discussed.

## Introduction

One often hopes to answer mathematical questions of existence (does this object or map possess a desired property?) without the necessity of construction. For it is undoubtedly advantageous to know whether a map possesses some extension or lift before making futile attempts at constructing one, or to know that a module is not finitely presented before we ever consider a possible finite presentation. Homological algebra offers a surprisingly useful tool to answer such questions by the way of cohomology: one can often define a sequence of cohomology classes whose elements, upon examination, possess the answer to the existence question initially posed. Most frequently such cohomology classes are defined to measure *obstructions* to answering the question: if the cohomology groups are all zero, we typically conclude that *yes*, the object or map possesses the desired property. On the other hand, if some cohomology group is nonzero, we obtain useful information about where and how exactly the property fails.

The functor $\mathrm{Ext}_R(M, -)$ is a prime example of using cohomology to measure obstructions, and is especially important in the context of local cohomology. Given $R$-modules $A$ and $B$, we ask whether there exists an *extension of $A$ by $B$*, that is, an $R$-module $C$ that fits into a short exact sequence of $R$-modules

$$0 \to B \to C \to A \to 0.$$

Such an extension always exists by taking $C = B \oplus A$ and using the canonical maps. It turns out that the elements of the first Ext group $\mathrm{Ext}^1_R(A, B)$ are in bijection with equivalence classes of extensions of $A$ by $B$, with the trivial extension $A \oplus B$ corresponding to the zero element of $\mathrm{Ext}^1_R(A, B)$. One then notices that the question "*is $A$ a projective $R$-module?*" is equivalent to checking that $\mathrm{Ext}^1_R(A, B) = 0$ for all $R$-modules $B$, for if

$$0 \to B \to C \to A \to 0$$

is exact then $A$ is projective if and only if the sequence splits. Hence one obtains an answer to a purely algebraic question by appealing to cohomology. On a similar note, if one already knows that $A$ is projective, then knowing that $\mathrm{Ext}^1_R(A, B) = 0$ for all $R$-modules $B$ allows us to easily compute $\mathrm{Ext}$ groups of closely related modules.

To motivate our foray into local cohomology we consider a question which served as part of Grothendieck's inspiration for introducing the construction in 1961 ([1]). Given a set of generators for a finitely generated ideal, it is natural to ask whether the cardinality of this set is minimal or if it is possible to yield the ideal with fewer generators. Alternatively one can pose this question "up to radical": given a ring $R$ and an ideal

$J$, what is the least number of elements $x_1, \dots, x_n$ of $J$ such that $\sqrt{(x_1, \dots, x_n)} = J$? In a purely constructive argument, we would need to try every possible combination of elements to definitively conclude that $J$ cannot be given with fewer generators up to radical. Hence the appeal of an obstruction which provides necessary lower bounds for this number of generators becomes clear, and Grothendieck's local cohomology is precisely the tool which accomplishes this.

In section 1 we provide the definition of local cohomology in terms of $\mathrm{Ext}$. Section 2 provides important consequences of this definition, focusing on vanishing theorems and theorems of uniqueness. In section 3 we compute some local cohomology groups of polynomial rings by making use of the vanishing theorems. The final section returns to the minimal radical generation question from the algebro-geometric point of view, discussing how local cohomology is used to solve this problem.

## 1 Defining Local Cohomology in Terms of $\mathrm{Ext}$

Let $R$ be a Noetherian ring, $M$ an $R$-module, $I \leq R$ an ideal. To define the $i^{th}$ local cohomology module of $M$ with support in $I$ we must first rigorously define the $\mathrm{Ext}$ functor and the corresponding $\mathrm{Ext}$ groups.

**Definition 1.** *Let $R$ be a ring and $N$ an $R$-module. A projective resolution of $N$ is an exact sequence*

$$\cdots \to P_1 \to P_0 \to N \to 0$$

*where each $P_i$ is a projective $R$-module.*

We remark that every $R$-module has a projective resolution via the following recipe: Take a surjective map from a projective module $P_0$ to $N$, which exists by taking $P_0$ to be the free group of which $N$ is a quotient. Consider the kernel $K_0$. There exists a surjective map from a projective module $P_1$ to $K_0$, so one obtains a map $P_1 \to P_0$ by composing $P_1 \twoheadrightarrow K_0 \hookrightarrow P_0$. The image of this map is precisely the kernel of $P_0 \to N$, yielding exactness at $P_0$. Letting $K_1 = \ker(P_1 \to K_0)$, we proceed inductively indefinitely.

We truncate this resolution by replacing $N$ with 0 to get a new sequence $P_\bullet = \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0$, which is no longer exact but which remains a chain complex ($\operatorname{im} d_i \subseteq \ker d_{i-1}$ for all $i \geq 1$). Given another $R$-module $M$, we may apply to $P_\bullet$ the left exact contravariant functor $\mathrm{Hom}_R(-, M)$, obtaining the cochain complex

$$0 \to \mathrm{Hom}_R(P_0, M) \xrightarrow{d_{1_*}} \mathrm{Hom}_R(P_1, M) \xrightarrow{d_{2_*}} \cdots \tag{1}$$

which often fails to be exact but which satisfies $\operatorname{im} d_{i_*} \subseteq \ker d_{i+1_*}$. The $i^{th}$ $\mathrm{Ext}$ group $\mathrm{Ext}^i_R(N, M)$ is the $i^{th}$ cohomology of (1):

**Definition 2.** *For the cochain complex in* (1),

$$\mathrm{Ext}^i_R(N, M) := \ker d_{i+1_*} / \operatorname{im} d_{i_*}.$$

*Thus $\mathrm{Ext}^i$ measures failure of exactness of* (1) *at the $i^{th}$ position.*

The definition of $\mathrm{Ext}^i_R(N, M)$ appears to rely heavily on the choice of projective resolution for $N$, so it is natural to ask whether $\mathrm{Ext}^i_R(N, M)$ is well-defined. Fortunately, given another projective resolution $Q_\bullet$ for $N$, the process outlined above yields isomorphic $\mathrm{Ext}$ groups to those already constructed (see [7]).

**Remark 1.** If we view $\mathrm{Hom}_R(-, M)$ as a right exact functor $\mathrm{Mod_R} \to \mathrm{Ab^{op}}$ instead of a contravariant left exact functor, then $\mathrm{Ext}^i_R(-, M)$ as defined above is the $i^{th}$ *left derived functor* of $\mathrm{Hom}_R(-, M)$. Alternatively it is possible (and often more useful) to take $\mathrm{Ext}^i_R(N, -)$ as the $i^{th}$ right derived functor of $\mathrm{Hom}_R(N, -)$. Hence $\mathrm{Ext}^i_R(N, M)$ is computed by taking an *injective resolution*

$$0 \to M \to I^0 \to I^1 \to \cdots$$

of $M$, truncating it to $I_\bullet$, and looking at the $i^{th}$ cohomology of

$$0 \to \mathrm{Hom}_R(N, I_0) \to \mathrm{Hom}_R(N, I_1) \to \cdots$$

For a proof that these two definitions are equivalent, we refer the reader to [7]. Each definition has its advantages: while injective resolutions tend to make for simpler proofs, projective resolutions are typically easier to construct. Hereon we use whichever is most convenient for the task at hand.

**Proposition 1.** $\mathrm{Ext}^i_R(-, M)$ is a contravariant functor from $\mathrm{Mod_R}$ to $\mathrm{Ab}$.

*Proof.* We have seen that $\mathrm{Ext}^i_R(-, M)$ maps $R$-modules to abelian groups. It remains to show that if $f : N \to N'$ is an $R$-module homomorphism, there is an induced group homomorphism $\tilde{f}_i : \mathrm{Ext}^i_R(N', M) \to \mathrm{Ext}^i_R(N, M)$ satisfying

(1) the induced map $\tilde{\mathrm{id}}_i : \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N, M)$ is the identity, and

(2) if $f : N \to N'$ and $g : N' \to N''$ are $R$-module homomorphisms, then $\widetilde{(g \circ f)}_i = \tilde{g}_i \circ \tilde{f}_i$.

We construct the induced map and leave the verification of (1) and (2) to the reader. Given a projective resolution

$$\cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} N' \to 0$$

for $N'$ and a projective resolution

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} N \to 0$$

for $N$, we define a chain map $h_* : P_* \to P'_*$ inductively: First we define $h_0 : P_0 \to P'_0$ so that the diagram

$$\begin{array}{ccc} P_0 & \overset{h_0}{\dashrightarrow} & P'_0 \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d'_0} \\ N & \xrightarrow{f} & N' \end{array}$$

commutes. Suppose $h_0, \ldots, h_{i-1}$ are defined. We then define $h_i : P_i \to P'_i$ so that

$$\begin{array}{ccc} P_i & \overset{h_i}{\dashrightarrow} & P'_i \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d'_i} \\ P_{i-1} & \xrightarrow{h_{i-1}} & P'_{i-1} \end{array}$$

commutes. Each $h_i$ naturally induces a map $h_{i*} : \mathrm{Hom}_R(P'_i, M) \to \mathrm{Hom}_R(P_i, M)$ given by precomposition with $h_i$. Similarly each $d_i$ induces a map $d_{i*} : \mathrm{Hom}_R(P_{i-1}, M) \to \mathrm{Hom}_R(P_i, M)$, and we obtain a diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P'_i, M) & \xrightarrow{h_{i*}} & \mathrm{Hom}_R(P_i, M) \\ {\scriptstyle d'_{i*}}\uparrow & & \uparrow{\scriptstyle d_{i*}} \\ \mathrm{Hom}_R(P'_{i-1}, M) & \xrightarrow{h_{i-1*}} & \mathrm{Hom}_R(P_{i-1}, M) \end{array}$$

that commutes. As $\operatorname{Hom}_R(-, M)$ is left-exact, we have a diagram of commuting squares and exact columns:

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow \\ \operatorname{Hom}_R(P_1', M) & \xrightarrow{h_{1*}} & \operatorname{Hom}_R(P_1, M) \\ {\scriptstyle d_{1*}'}\uparrow & & {\scriptstyle d_{1*}}\uparrow \\ \operatorname{Hom}_R(P_0', M) & \xrightarrow{h_{0*}} & \operatorname{Hom}_R(P_0, M) \end{array}$$

In particular we have a map

$$\begin{aligned} \operatorname{Ext}_R^i(N', M) &\to \operatorname{Ext}_R^i(N, M) \\ [x] &\mapsto \overline{h_{i*}(x)} \end{aligned}$$

where $x \in \ker d_{i+1*}'$, $[x]$ refers to an equivalence class of $\ker d_{i+1*}' / \operatorname{im} d_{i*}'$, and $\overline{h_{i*}(x)}$ refers to an equivalence class of $\ker d_{i+1*} / \operatorname{im} d_{i*}$. Certainly if $x \in \ker d_{i+1*}'$ then $h_{i*}(x) \in \ker d_{i+1*}$, since $d_{i+1*} \circ h_{i*}(x) = h_{i+1*} \circ d_{i+1*}'(x) = h_{i*}(0) = 0$. Furthermore this map is well defined since if $[x] = [y]$ then $x - y \in \operatorname{im} d_{i*}'$, and thus there exists some $a \in \operatorname{Hom}_R(P_{i-1}', M)$ such that $d_{i*}'(a) = x - y$. Then $h_{i*}(d_{i*}'(a)) = d_{i*}(h_{i-1*}(a))$, which implies $h_{i*}(x) - h_*(y) \in \operatorname{im} d_{i*}$. We conclude that $\overline{h_{i*}(x)} = \overline{h_{i*}(y)}$. □

**Corollary 1.** Let $R$ be a Noetherian ring with $I \leq R$ an ideal, $M$ be an $R$-module. For each $k \in \mathbb{N}$, the surjection $R/I^{k+1} \twoheadrightarrow R/I^k$ induces a map on the $i^{th}$ Ext group

$$\operatorname{Ext}_R^i(R/I^k, M) \xrightarrow{g_k} \operatorname{Ext}_R^i(R/I^{k+1}, M).$$

To obtain our $i^{th}$ local cohomology group we need the notion of a *direct limit.*

**Definition 3.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of groups and let $g_n : G_n \to G_{n+1}$ be a group homomorphism for each $n$. Let $G_\infty$ be a group and for each $n$ let $u_n : G_n \to G_\infty$ be a group homomorphism. We say $(G_\infty, (u_n))$ is the <u>direct limit</u> of $(G_n, g_n)$ if*

*(1) for all $n \in \mathbb{N}$,*

$$\begin{array}{ccc} G_n & \xrightarrow{g_n} & G_{n+1} \\ & {\scriptstyle u_n}\searrow & \downarrow{\scriptstyle u_{n+1}} \\ & & G_\infty \end{array}$$

*commutes, and*

*(2) for any group $H$ and group homomorphisms $h_n : G_n \to H$ satisfying commutativity with the $g_n$ as above, we have a unique group homomorphism $h_\infty : G_\infty \to H$ so that*

$$\begin{array}{ccc} G_n & \xrightarrow{h_n} & H \\ & {\scriptstyle u_n}\searrow & \uparrow{\scriptstyle h_\infty} \\ & & G_\infty \end{array}$$

*commutes for each $n$.*

**Remark 2.** If we partially order finitely generated submodules of an $R$-module $M$ by inclusion, the direct limit $\varinjlim_i M_i$ of the collection of all finitely generated submodules of $M$ is $\bigcup_i M_i = M$.

**Definition 4.** *Let $R$ be a Noetherian ring with $I \leq R$ an ideal of $R$, and let $M$ be an $R$-module. The* *$i^{th}$ local cohomology of $M$ with support in $I$*, *denoted $H^i_I(M)$, is the direct limit of $(\operatorname{Ext}^i_R(R/I^k, M), g_k)$, where*

$$g_k : \operatorname{Ext}^i_R(R/I^k, M) \to \operatorname{Ext}^i_R(R/I^{k+1}, M)$$

*is the map induced by the surjection $R/I^{k+1} \twoheadrightarrow R/I^k$. We write $H^i_I(M) = \varinjlim_k \operatorname{Ext}^i_R(R/I^k, M)$.*

A simple consequence of the definition is that every local cohomology module $H^i_I(M)$ is $I$-torsion.

**Proposition 2.** Let $R$ be a Noetherian ring with $I \leq R$, and let $M$ be any $R$-module. Every element of $H^i_I(M)$ is annihilated by a power of $I$ ([4]).

*Proof.* Since $H^i_I(M)$ is defined to be the direct limit $\varinjlim_t \operatorname{Ext}^i_R(R/I^t, M)$, then every element of $H^i_I(M)$ is in the image of some $\operatorname{Ext}^i_R(R/I^t, M)$ for some $t$. But $\operatorname{Ext}^i_R(R/I^t, M)$ is killed by $I^t$, since $\operatorname{Hom}(R/I^t, J)$ is killed by $I^t$ for any injective module $J$, and Remark 1 allows us to compute $\operatorname{Ext}^i_R(R/I^t, M)$ via injective resolutions for $M$. □

For general $i$ it can be difficult to understand the elements of $H^i_I(M)$. In the case $i = 0$ there is a simple and useful characterization involving the *$I$-torsion functor* $\Gamma_I$ ([4]):

**Proposition 3.** $\Gamma_I(M) := H^0_I(M) = \{x \in M : I^t x = 0 \text{ for some } t \in \mathbb{N}\} = \bigcup_t \operatorname{Ann}_M I^t$, where $\operatorname{Ann}_M I^t = \{x \in M : I^t x = 0\}$.

*Proof.* We compute the direct limit of $\operatorname{Ext}^0_R(R/I^t, M)$. By definition $\operatorname{Ext}^0_R(R/I^t, M)$ is the kernel of $\operatorname{Hom}(P_0, M) \xrightarrow{d_1^*} \operatorname{Hom}(P_1, M)$ where $\cdots \to P_1 \xrightarrow{d_1} P_0 \to R/I^t \to 0$ is a projective resolution of $R/I^t$. By left exactness of $\operatorname{Hom}_R(-, M)$ we have

$$0 \to \operatorname{Hom}_R(R/I^t, M) \to \operatorname{Hom}_R(P_0, M) \xrightarrow{d_{1*}} \operatorname{Hom}_R(P_1, M)$$

is exact and thus $\operatorname{Ext}^0_R(R/I^t, M) = \ker d_{1*} \cong \operatorname{Hom}_R(R/I^t, M)$. Since an $R$-module homomorphism $\varphi : R/I^t \to M$ is determined by where it sends $\overline{1}$, then if $\varphi(\overline{1}) = m$ we know $0 = \varphi(\overline{i}) = im$ for all $i \in I^t$, so $m \in \operatorname{Ann}_M I^t$. Likewise for each $m \in \operatorname{Ann}_M I^t$ we may construct an $R$-module homomorphism $\varphi : R/I^t \to M$ such that $\varphi(\overline{1}) = m$, so we identify $\operatorname{Hom}(R/I^t, M) \cong \operatorname{Ann}_M I^t$. Thus for each $t \in \mathbb{N}$ the induced map

$$\operatorname{Ext}^0_R(R/I^t, M) \to \operatorname{Ext}^0_R(R/I^{t+1}, M)$$

is identified with the inclusion $\operatorname{Ann}_M I^t \hookrightarrow \operatorname{Ann}_M I^{t+1}$. It remains to show $(\bigcup_t \operatorname{Ann}_M I^t, (u_t)_t)$ is the direct limit of $\operatorname{Ann}_M I^t \xhookrightarrow{\iota_t} \operatorname{Ann}_M I^{t+1}$, where for each $t$ the map $u_t : \operatorname{Ann}_M I^t \to \bigcup_t \operatorname{Ann}_M I^t$ is given by inclusion. The diagram

$$\begin{array}{ccc} \operatorname{Ann}_M I^t & \xrightarrow{\iota_t} & \operatorname{Ann}_M I^{t+1} \\ & {\scriptstyle u_t}\searrow & \downarrow{\scriptstyle u_{t+1}} \\ & & \bigcup_t \operatorname{Ann}_M I^t \end{array}$$

commutes for all $t$ since all maps are inclusions. Furthermore if $H$ is a group and for all $t$ we have $h_t : \operatorname{Ann}_M I^t \to H$ commutes with the inclusion maps $\iota_t$, then the map $h_\infty : \bigcup_t \operatorname{Ann}_M I^t \to H$, $m \mapsto h_t(m)$ (where $t$ is such that $m \in \operatorname{Ann}_M I^t$) is the unique group homomorphism which makes

$$\begin{array}{ccc} \operatorname{Ann}_M I^t & \xrightarrow{h_t} & H \\ & {\scriptstyle u_t}\searrow & \uparrow{\scriptstyle h_\infty} \\ & & \bigcup_t \operatorname{Ann}_M I^t \end{array}$$

commute. Indeed this homomorphism is well-defined since if $m \in \operatorname{Ann}_M I^{t_1} \cap \operatorname{Ann}_M I^{t_2}$ for $t_1 \leq t_2$, then $h_{t_1}(m) = h_{t_2}(m)$, as the $h_t$ maps commute with inclusions. Hence $H_I^0(M) = \bigcup_t \operatorname{Ann}_M I^t$. ☐

**Remark 3.** In the proof of Proposition 3 we showed that $\Gamma_I(-)$ is equivalent to the functor $\varinjlim_t \operatorname{Hom}_R(R/I^t, -)$. By Remark 1 we know that $\operatorname{Ext}_R^i(R/I^t, -)$ is the $i^{th}$ right derived functor of the (left-exact) functor $\operatorname{Hom}(R/I^t, -)$. In this case, being a right derived functor commutes with direct limits, so $H_I^i(-) = \varinjlim_t \operatorname{Ext}_R^i(R/I^t, -)$ is actually the $i^{th}$ right derived functor of $\Gamma_I(-) = \varinjlim_t \operatorname{Hom}(R/I^t, -)$ ([5]). This equivalent characterization allows us to deduce facts about local cohomology from knowledge of $\Gamma_I$.

## 2 Vanishing Theorems and Uniqueness

Let $R$ be Noetherian and let $M$ be an $R$-module. To every ideal $I \leq R$ we have just associated a sequence of abelian groups $H_I^i(M)$. It is natural to ask whether this sequence is uniquely encoded by $I$, or if distinct ideals can yield the same local cohomology groups. It turns out that uniqueness holds *up to radical*:

**Proposition 4** (Local cohomology is unique up to radical ([2]))**.** Given ideals $I, J$ of a Noetherian ring $R$,

$$\sqrt{I} = \sqrt{J} \implies H_I^i(M) \cong H_J^i(M).$$

for any $R$-module $M$.

*Proof.* Suppose that $\sqrt{I} = \sqrt{J}$. Then $I \subseteq \sqrt{J}$ and $I$ finitely generated implies there exists some $t \in \mathbb{N}$ such that $I^t \subseteq J$. Similarly there exists some $s \in \mathbb{N}$ such that $J^s \subseteq I$. This means that for every element of $\{I^k\}_k$ there exists some element of $\{J^\ell\}_\ell$ with $I^k \subseteq J^\ell$ and similarly for every element of $\{J^k\}_k$ there exists some element of $\{I^\ell\}_\ell$ with $J^k \subseteq I^\ell$. Sets which satisfy such a property are said to be *cofinal* with respect to each other. Cofinality and Proposition 3 imply that $\Gamma_I(M) = \Gamma_J(M)$. Hence the right derived functors of $\Gamma_I$ and $\Gamma_J$ are equivalent, so by Remark 3 we know $H_I^i(M) \cong H_J^i(M)$ for all $i$. ☐

**Remark 4.** The converse of Proposition 3 holds if the functors $H_I^i$ and $H_J^i$ are equivalent for all $i$ ([2]). That is, if $H_I^i(M) \cong H_J^i(M)$ for every $R$-module $M$ and $i \in \mathbb{N} \cup \{0\}$, then $\sqrt{I} = \sqrt{J}$.

Since cohomology is used to measure obstructions, it is helpful to know when local cohomology is certain to vanish. The following theorem guarantees that local cohomology vanishes for $i > t$ whenever $I$ can be generated by $t$ elements, thus giving a vanishing condition independent of the module $M$.

**Proposition 5.** Suppose $I$ is generated by $t$ elements. For every $R$-module $M$ we have $H_I^i(M) = 0$ for all $i > t$.

*Proof.* As in [5], we proceed by induction on $t$. When $t = 0$ this means $I = 0$, so $\Gamma_0(M) = H_0^0(M) = \{x \in M : 0x = 0\} = M$. Hence $\Gamma_0$ is the identity functor, so by our definition of local cohomology as the right derived functor of $\Gamma$ we conclude that $H_0^i(M) = 0$ for all $i > 0$. When $t = 1$ we have that $I$ is principal, so we write $I = (a)$ for $a \neq 0$ and notice that $I^t = (a^t)$ for all $t$. If $a$ is not a zero divisor then we have a short exact sequence of $R$-modules

$$0 \to R \xrightarrow{\cdot a^t} R \twoheadrightarrow R/(a^t) \to 0.$$

It is a standard fact of the theory of derived functors that short exact sequences induce long exact sequences of cohomology ([7]), so the short exact sequence above induces a long exact sequence of Ext groups

$$\cdots \to \operatorname{Ext}_R^1(R/(a^t), M) \to \operatorname{Ext}_R^1(R, M) \to \operatorname{Ext}_R^1(R, M) \to \operatorname{Ext}_R^2(R/(a^t), M) \to \operatorname{Ext}_R^2(R, M) \to \cdots$$

which is equivalent to

$$\cdots \to \operatorname{Ext}^1_R(R/(a^t), M) \to 0 \to 0 \to \operatorname{Ext}^2_R(R/(a^t), M) \to 0 \to \cdots$$

due to $R$ being a free (hence projective) $R$-module, which immediately implies $\operatorname{Ext}^i_R(R, M) = 0$ for $i > 0$. We see then that $\operatorname{Ext}^i_R(R/(a^t), M) = 0$ for all $i > 1$ and $t \in \mathbb{N}$, so necessarily $H^i_{(a)}(M) = 0$ for all $i > 1$. If $a$ is a zero-divisor of $R$ the result still holds but the proof is not as immediate: By Theorem 2.2.16 in [5], we have a natural equivalence of functors

$$\omega : D_{(a)}(-) = \varinjlim_n \operatorname{Hom}_R((a^n), -) \cong (-)_a$$

where $M_a$ is the localization of $M$ with respect to the multiplicative set $\{1, a, a^2, \dots\}$, and $\omega$ maps $f \in \operatorname{Hom}_R((a^t), M)$ to $\frac{f(a^t)}{a^t} \in M_a$. Since localizations are exact, we see that $D_{(a)}(-) = \varinjlim_n \operatorname{Hom}_R((a^n), -)$ is an exact functor, and thus the right derived functors $\mathscr{R}^i D_{(a)}(-)$ are trivial for $i > 0$. By Theorem 2.2.4 in [5], $\mathscr{R}^i D_{(a)}(-) \cong H^{i+1}_{(a)}(-)$, so we immediately obtain $H^i_{(a)}(M) = 0$ for all $i > 1$ and $R$-modules $M$.

Now suppose that $t > 1$ and the result is true for ideals generated by $t$ elements or fewer. If $I \leq R$ is generated by $t+1$ elements $x_1, \dots, x_{t+1}$, we consider the ideals $\mathfrak{a} = (x_1, \dots, x_t)$ and $\mathfrak{b} = (x_{t+1})$, so that $I = \mathfrak{a} \oplus \mathfrak{b}$. Our inductive assumption lets us conclude that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > t$ and that $H^i_{\mathfrak{b}}(M) = 0$ for all $i > 1$. To complete the proof we need a way to express $H^i_I(M)$ in terms of $H^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{b}}(M)$. This is done via a *Mayer-Vietoris sequence.*

**Lemma 1** (Mayer-Vietoris for Local Cohomology)**.** *For each $R$-module $M$ and ideals $\mathfrak{a}, \mathfrak{b} \leq R$, there is a long exact sequence of local cohomology*

$$0 \longrightarrow H^0_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \longrightarrow H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow H^1_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) \longrightarrow H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \cdots$$

See Theorem 3.2.3 of [5] for a proof. For all $i > t+1$ we know that $H^i_{\mathfrak{a}}(M) = 0$ and $H^i_{\mathfrak{b}}(M) = 0$, so exactness of the Mayer-Vietoris sequence yields an exact sequence

$$0 \to H^{i-1}_{\mathfrak{a}\cap\mathfrak{b}}(M) \to H^i_I(M) \to 0$$

for all $i > t+1$, where we recall that $I = \mathfrak{a} + \mathfrak{b}$. Thus $H^i_I(M) \cong H^{i-1}_{\mathfrak{a}\cap\mathfrak{b}}(M)$. Since $\sqrt{\mathfrak{a}\cap\mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}$, Proposition 3 tells us that $H^i_{\mathfrak{a}\mathfrak{b}}(M) \cong H^i_{\mathfrak{a}\cap\mathfrak{b}}(M)$ for all $i$. Recall that $\mathfrak{a}$ is generated by $x_1, \dots, x_t$ and that $\mathfrak{b}$ is generated by $x_{t+1}$, so $\mathfrak{a}\mathfrak{b}$ is generated by the $t$ elements $x_1x_{t+1}, \dots, x_tx_{t+1}$. By our inductive hypothesis, we know that $H^i_{\mathfrak{a}\mathfrak{b}}(M) = 0$ for all $i > t$, or equivalently $H^{i-1}_{\mathfrak{a}\mathfrak{b}}(M) = 0$ for all $i > t+1$. We conclude that $H^i_I(M) = 0$ for all $i > t+1$. □

The main theorem of this section provides a vanishing condition based solely on the module $M$ and independent of the chosen support ideal $I$. The proof of the theorem requires several auxiliary results, the first of which concerns vanishing of local cohomology when the module is annihilated by powers of the support ideal. The second discusses the contrary case, when $M$ is $I$-torsion-free.

**Lemma 2.** *If $M$ is an $I$-torsion module, then $H^i_I(M) = 0$ for all $i > 0$ [5].*

*Proof.* If $M$ is $I$-torsion, we can always find an injective resolution $J_\bullet$ of $M$ consisting of $I$-torsion modules ([5], Corollary 2.1.6). If $J_i$ is an injective $I$-torsion module, then the exact sequence

$$0 \to I^2 \to R \to R/I^2 \to 0$$

yields an exact sequence

$$0 \to \mathrm{Hom}_R(R/I^2, J_i) \to \mathrm{Hom}_R(R, J_i) \to \mathrm{Hom}_R(I^2, J_i) \to 0.$$

We note that $\mathrm{Hom}_R(I^2, J_i) = 0$, since if $f \in \mathrm{Hom}_R(I^2, J_i)$ then for all $i, j \in I$ we have $f(ij) = if(j) = 0$ as $J_i$ is $I$-torsion. Therefore $\mathrm{Hom}_R(R/I^2, J_i) \cong \mathrm{Hom}_R(R, J_i) \cong J_i$, so our chain complex

$$0 \to \mathrm{Hom}_R(R/I^2, J_0) \to \mathrm{Hom}_R(R/I^2, J_1) \to \cdots$$

becomes

$$0 \to J_0 \to J_1 \to \cdots$$

which is exact at all $J_i$ for $i > 0$ by exactness of the original injective resolution. Therefore $\mathrm{Ext}^i_R(R/I^2, M) = 0$ for all $i > 0$. An inductive argument shows that $\mathrm{Hom}_R(R/I^t, J_i) \cong J_i$ for all $t > 2$ as well, so that $\mathrm{Ext}^i_R(R/I^t, M) = 0$ for all $t \geq 2$. In the direct limit we conclude $H^i_I(M) = 0$ for $i > 0$. $\square$

**Lemma 3.** *Let $M$ be a finitely generated $R$-module. If $M$ is $I$-torsion free, then $I$ contains a non-zerodivisor on $M$ [5].*

*Proof.* We show the contrapositive. Suppose $I$ consists entirely of zero divisors of $M$. Since the set of zero divisors of $M$ is equal to the union of the associated primes of $M$ (where $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if there exists some nonzero $x \in M$ such that $\mathrm{Ann}_R(x) = \mathfrak{p}$), we have $I \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$. By the Prime Avoidance Lemma we get $I \leq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(M)$, and since $\mathfrak{p} = \mathrm{Ann}_R(x)$ for some $x \in M$, then $I$ annihilates $x \in M$. Therefore $x \in \Gamma_I(M)$ and we conclude that $M$ is *not* $I$-torsion free. $\square$

To prove Grothendieck's Vanishing Theorem we need the key fact that *flat base change* commutes with local cohomology:

**Lemma 4.** *If $R$ is Noetherian and $S'$ is a flat Noetherian $R$-algebra, then*

$$S' \otimes_R H^i_I(M) \cong H^i_I(S' \otimes_R M) \cong H^i_{IS'}(S' \otimes_R M)$$

*as $S'$-modules ([4]).*

As localizations of a Noetherian ring $R$ are $R$-flat and Noetherian, we obtain as a consequence of Lemma 4 that if $S$ is a multiplicatively closed subset of $R$, then

$$S^{-1}(H^i_I(M)) \cong H^i_{S^{-1}I}(S^{-1}M)$$

as $S^{-1}R$-modules.

### 2.1 Grothendieck's Vanishing Theorem

**Theorem 1** (Grothendieck's Vanishing Theorem)**.** *Let $R$ be Noetherian, $I \leq R$, $M$ an $R$-module. Let $\dim_R(M)$ be the Krull dimension of $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : M_\mathfrak{p} \neq 0\}$. Then $H^i_I(M) = 0$ for all $i > \dim_R(M)$ ([3]).*

*Proof.* It suffices to prove the theorem in the case where $M$ is finitely generated: this follows from the fact that $M$ is a direct limit of its finitely generated submodules by Remark 2, that local cohomology commutes with direct limits ([4]), and that the dimension of a submodule of $M$ cannot exceed the dimension of $M$.

Since being zero is a local property of modules, it suffices to show that $(H^i_I(M))_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \in \mathrm{Spec}(R), i > \dim_R(M)$. By Lemma 4 we know that localizations commute with local cohomology, and thus the problem reduces to showing $H^i_{IR_\mathfrak{m}}(M_\mathfrak{m}) = 0$ for every maximal $\mathfrak{m} \in \operatorname{Spec}(R)$, $i > \dim_R(M)$. Fix $\mathfrak{m} \in \operatorname{Spec}(R)$. We claim that $\dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \leq \dim_R(M)$, so that it suffices to prove the proposition in the case where $(R, \mathfrak{m})$ is a local ring. Indeed, if we can show that

$$\operatorname{Supp}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \{\mathfrak{q}R_\mathfrak{m} : \mathfrak{q} \in \operatorname{Supp}(M) \text{ and } \mathfrak{q} \subseteq \mathfrak{m}\},$$

then any chain of prime ideals in the support of $M_\mathfrak{m}$ corresponds to a chain of prime ideals contained in $\mathfrak{m}$ in the support of $M$, and hence cannot be of length greater than a general chain of prime ideals in the support of $M$. To prove the equality above, we note that if $\mathfrak{q}$ is a prime in $R$ such that $\mathfrak{q} \leq \mathfrak{m}$ and $M_\mathfrak{q} \neq 0$, then the extended ideal $\mathfrak{q}R_\mathfrak{m} \in \operatorname{Spec}(R_\mathfrak{m})$ is in the support of $M_\mathfrak{m}$, as

$$\begin{aligned}(M_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}} \cong (M \otimes_R R_\mathfrak{m}) \otimes_{R_\mathfrak{m}} (R_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}} &\cong M \otimes_R (R_\mathfrak{m} \otimes_{R_\mathfrak{m}} (R_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}}) \\ &\cong M \otimes_R (R_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}}.\end{aligned}$$

But $(R_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}} \cong R_\mathfrak{q}$ via $\frac{(\frac{r}{s})}{(\frac{r'}{s'})} \mapsto \frac{rs'}{r's}$, since $r's \notin \mathfrak{q}$ as both $r' \notin \mathfrak{q}$ and $s \notin \mathfrak{m} \supseteq \mathfrak{q}$. This means that $(M_\mathfrak{m})_{\mathfrak{q}R_\mathfrak{m}} \cong M \otimes_R R_\mathfrak{q} \cong M_\mathfrak{q}$ which is nonzero. On the other hand, if $\mathfrak{q}R_\mathfrak{m}$ is in the support of $M_\mathfrak{m}$, then $\mathfrak{q}$ is necessarily contained in $\mathfrak{m}$ (for $\mathfrak{q}R_\mathfrak{m}$ to even be a prime ideal of $R_\mathfrak{m}$) and the same isomorphism above yields $M_\mathfrak{q} \neq 0$.

We have thus reduced to the local case. By convention if $M = 0$ then $\dim_R(M) = -1$, so certainly $H^i_I(0) = 0$ for all $i > -1$. On the other hand if $I = R$ is the unit ideal then $\Gamma_I$ is the zero functor, for $\Gamma_I(M) = \Gamma_R(M) = \{x \in M : Rx = 0\} = 0$. Since the higher local cohomology groups are right derived functors of $\Gamma_I$, they must be zero. It remains to prove the theorem when $M \neq 0$ and $I < \mathfrak{m}$ properly.

Suppose $\dim_R(M) = 0$. For finitely generated modules we have $\operatorname{Supp}(M) = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal $\operatorname{Ann}_R(M) = \{r \in R : rm = 0 \text{ for some } m \in M\}$, and thus $\dim_R(M) = \dim(\operatorname{Supp}(M)) = \dim(R/\mathfrak{a})$. We conclude that $\dim(R/\operatorname{Ann}_R M) = 0$, and since $R/\operatorname{Ann}_R M$ is Noetherian as a quotient of a Noetherian ring, its dimension then forces it to be Artinian. Since every prime ideal is maximal and $R$ is local by assumption, some power of the maximal ideal $\overline{\mathfrak{m}}^n = \overline{0} = \operatorname{Ann}_R M$, so every element of $M$ is annihilated by some power of $\mathfrak{m}$. As an immediate consequence we have that every element of $M$ is annihilated by a power of any proper ideal $I < \mathfrak{m}$, so $M$ is $I$-torsion. By Proposition 2, $H^i_I(M) = 0$ for all $i > 0 = \dim_R(M)$, completing the base case.

Now suppose the statement holds for all modules of dimension smaller than $n$. Since a short exact sequence of $R$-modules induces a long exact sequence on local cohomology ([5]), we consider the long exact sequence of local cohomology induced by the short exact sequence

$$0 \to \Gamma_I(M) \hookrightarrow M \twoheadrightarrow M/\Gamma_I(M) \to 0.$$

As $H^i_I(\Gamma_I(M)) = 0$ for all $i > 0$ (since $\Gamma_I(M)$ is $I$-torsion by definition), then $H^i_I(M) \cong H^i_I(M/\Gamma_I(M))$ for all $i > 0$. Now $M/\Gamma_I(M)$ is $I$-torsion free and has dimension at most the dimension of $M$, so hereafter we assume without loss of generality that $M$ is $I$-torsion free. Lemma 3 implies that there exists some $r \in I$ which is a non-zerodivisor on $M$. For $i > n$ and every $t \in \mathbb{N}$ we consider the exact sequence

$$0 \to M \xrightarrow{\cdot r^t} M \twoheadrightarrow M/r^t M \to 0$$

which induces an exact sequence $H^{i-1}_I(M/r^tM) \to H^i_I(M) \to H^i_I(M)$. The second map in this sequence is again just multiplication by $r^t$, since the functor $H^i_I$ is $R$-linear as a consequence of being a right derived functor of the $R$-linear functor $\Gamma_I$. As $r$ is a non-zerodivisor on $M$, we know that $r^t \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$ for any $t \in \mathbb{N}$, and since it is a fact in commutative algebra that minimal elements of $\operatorname{Supp}(M)$ belong to $\operatorname{Ass}(M)$, we know $r^t \notin \mathfrak{p}$ for some minimal $\mathfrak{p} \in \operatorname{Supp}(M)$. Then $\mathfrak{p} \notin \operatorname{Supp}(M/r^tM)$, since $(M/r^tM)_\mathfrak{p} \cong$

$M_\mathfrak{p}/r^t M_\mathfrak{p}$ and $r^t M_\mathfrak{p} \cong M_\mathfrak{p}$ due to the fact that $r^t$ is invertible in $M_\mathfrak{p}$. This implies that $\dim(M/r^t M)$ is strictly less than $\dim(M)$, and therefore $H_I^{i-1}(M/r^t M) = 0$ for all $i - 1 > n - 1$ by the inductive hypothesis. Hence for $i > n$ and every $t \in \mathbb{N}$ we have

$$H_I^i(M) \xhookrightarrow{\cdot r^t} H_I^i(M)$$

is an injection. But $H_I^i(M)$ is $I$-torsion by Proposition 2, and since $r \in I$ there must exist some $t \in \mathbb{N}$ for which $r^t(H_I^i(M)) = 0$. Injectivity then forces $H_I^i(M) = 0$. □

## 3 Computing Examples: Polynomial Rings

Just as it was useful to know when local cohomology certainly vanishes, it is helpful to know when we should expect some nonzero local cohomology, especially in the context of Proposition 5 which is concerned with minimal generation of ideals. The following lemma gives a sufficient and necessary condition for nonvanishing local cohomology.

**Lemma 5.** *([5]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring $R$ generated by $a_1, \ldots, a_n$. Then $H_\mathfrak{a}^n(R) \neq 0$ if and only if there exists some $k \in \mathbb{N}$ such that, for every $t \in \mathbb{N}$,*

$$(a_1, \ldots, a_n)^t \not\subseteq Ra_1^{t+k} + \cdots + Ra_n^{t+k}.$$

We use the preceding lemma to prove several facts about polynomial rings.

**Example 1.** Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $k$, and let $I = (x_1, \ldots, x_n)$ be the ideal of polynomials with zero constant term. By Proposition 5, we have that for any $R$-module $M$ and every $i > n$, $H_I^i(M) = 0$. Notice that for all $t \in \mathbb{N}$ we have

$$(x_1, \ldots, x_n)^t \not\subseteq Rx_1^{t+1} + \cdots + Rx_n^{t+1}$$

as, for instance, $x_i^t \notin (x_1^{t+1}, \ldots, x_n^{t+1})$ for any $1 \leq i \leq n$. Lemma 5 applies with $k = 1$ to allow us to conclude that it is not possible to generate $I$ with fewer than $n$ generators.

When $R = k[x]$ and $I = (x)$, Proposition 5 implies that $H_{(x)}^i(M) = 0$ for every $i > 1$ and $k[x]$-module $M$. It is illustrative to compute $H_{(x)}^0(M)$ and $H_{(x)}^1(M)$ for general (finitely generated) $M$ in this setting.

**Example 2.** ([1]) Let $R = k[x]$, $I = (x)$, and $M$ a finitely generated $R$-module. By the structure theorem for finitely generated modules over principal ideal domains, $M \cong k[x]^n \bigoplus_{i=1}^{\ell} k[x]/(f_i^{n_i})$, where $n, \ell \in \mathbb{N} \cup \{0\}$, $f_i$ is an irreducible polynomial in $k[x]$, and $n_i \in \mathbb{N}$. Let $k_i$ be the largest nonnegative power of $x$ dividing $f_i^{n_i}$. Then

$$H_I^0(M) \cong \bigoplus_{i=1}^{\ell} k[x]/(x^{k_i})$$
$$H_I^1(M) \cong \bigoplus_{i=1}^{n} k[x, x^{-1}]/k[x]$$

*Proof.* We first compute $H_I^0(M)$. As Ext commutes with finite direct sums in the second component, and direct limits commute with direct sums ([1]), it is enough to compute $H_I^0(k[x])$ and $H_I^0(k[x]/(f_i^{n_i})$ separately. Since $H_I^0(k[x]) \cong \Gamma_I(k[x])$ and no nonzero elements of $k[x]$ are annihilated by any power of

$(x)$, we obtain $H^0_I(k[x]) = 0$ by Proposition 3. To compute $H^0_I(k[x]/(f_i^{n_i})$ for $f_i$ irreducible and nonzero, we consider the short exact sequence

$$0 \to k[x] \xrightarrow{\cdot f_i^{n_i}} k[x] \twoheadrightarrow k[x]/(f_i^{n_i}) \to 0$$

which induces a long exact sequence

$$0 \to H^0_I(k[x]/(f_i^{n_i})) \to H^1_I(k[x]) \xrightarrow{\cdot f_i^{n_i}} H^1_I(k[x]) \to H^1_I(k[x]/(f_i^{n_i})) \to H^2_I(k[x]) \to \cdots$$

with the multiplication by $f_i^{n_i}$ map carrying over to local cohomology by virtue of $H^i_I$ being $k[x]$-linear, as remarked in the proof of Theorem 1. For both the computation of $H^0_I(k[x]/(f_i^{n_i})$ and the eventual computation of $H^1_I(M)$ it is apparent we need to understand $H^1_I(k[x])$. Take an injective resolution

$$0 \to k[x] \to k(x) \to k(x)/k[x] \to 0$$

where $k(x)$ is the field of rational functions. Applying the exact functor $\Gamma_I$ yields an exact sequence

$$0 \to \Gamma_{(x)}(k(x)) \to \Gamma_{(x)}(k(x)/k[x]) \to 0$$

since $\Gamma_{(x)}(k[x]) = 0$ as $k[x]$ is $(x)$-torsion free. Then $\Gamma_{(x)}(k(x)/k[x]) \cong k[x, x^{-1}]/k[x]$, as the only way elements in $k(x)/k[x]$ are annihilated by powers of $(x)$ is if they are of the form $\frac{f(x)}{x^n}$ for some $f(x) \in k[x]$ and $n \in \mathbb{N}$. Because $H^1_I(k[x]) \cong \Gamma_{(x)}(k(x)/k[x])$, we conclude that $H^1_{(x)}(k[x]) \cong k[x, x^{-1}]/k[x]$.

Now $H^0_I(k[x]/(f_i^{n_i}))$ is isomorphic to the kernel of

$$k[x, x^{-1}]/k[x] \xrightarrow{\cdot f_i^{n_i}} k[x, x^{-1}]/k[x].$$

Expressing $f_i^{n_i}$ as $x^{k_i}g$ where $k_i \in \mathbb{N} \cup \{0\}$ and $x \nmid g$, we see that an element $\overline{h(x)}$ of $H^1_{(x)}(k[x]) \cong k[x, x^{-1}]/k[x]$ is in the kernel of $\cdot f_i^{n_i}$ only if it is some $k[x]$ multiple of $\frac{1}{x^{k_i}}$, and thus

$$H^0_I(k[x]/f_i^{n_i}) \cong k[x](\frac{1}{x^{k_i}})/k[x] \cong k[x]/(x^{k_i}).$$

This completes our calculation of $H^0_{(x)}(M)$.

It remains to compute $H^1_{(x)}(k[x]/(f_i^{n_i}))$. Since $H^2_{(x)}(k[x]) = 0$, exactness of the long exact sequence implies that $H^1_{(x)}(k[x]/(f_i^{n_i})$ is the cokernel of $H^1_{(x)}(k[x]) \xrightarrow{\cdot f_i^{n_i}} H^1_{(x)}(k[x])$. We claim that this map is surjective, which holds if $H^1_{(x)}(k[x])$ is a divisible $k[x]$-module. Factoring $f_i^{n_i} = x^{k_i}g$ as before, we note that multiplication by $x^{k_i}$ is certainly surjective, and it remains to show that multiplication by $g$ is as well. Since $g$ is not divisible by $x$ then either $g \in k$ (in which case surjectivity is again trivial), or $g$ is of the form $a_0 + p(x)$ where $a_0 \in k^\times$. As $p(x)$ is nilpotent in $H^1_{(x)}(k[x])$, the sum of $p(x)$ and $a_0$ must be a unit in $H^1_{(x)}(k[x])$, and multiplication by this unit is surjective. We conclude that $\operatorname{coker}(\cdot f_i^{n_i}) = H^1_{(x)}(k[x]/(f_i^{n_i})) = 0$, completing the case of $H^1_{(x)}(M)$. □

# 4 Minimal Generation of Radicals

We conclude the exposition by briefly returning to the question posed in the introduction: given a ring $R$ and an ideal $J$, what is the least number of elements $x_1, \ldots, x_n$ of $J$ whose radical $\sqrt{(x_1, \ldots, x_n)}$ generates $J$? This question has important roots in the theory of *complete intersections* in algebraic geometry, where one asks whether the vanishing ideal $I(C)$ of a curve $C$ in projective $n$ space can be generated by $r$ elements, where $r = \operatorname{codim}_{\mathbb{P}^n} C$. In other words, given an algebraic variety $C = V(f_1, \ldots, f_k)$ where each $f_i$ is a homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ and $C$ has dimension $m$ in $\mathbb{P}^n$, we wish to determine whether there exist $n - m$ homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ which generate every other homogeneous polynomial that vanishes on $C$. Since Hilbert's Nullstellensatz tells us that $I(V(J)) = \sqrt{J}$, it is enough to consider whether there exist $n-m$ homogeneous polynomials whose *radical* generates the vanishing ideal ([6]).

**Definition 5.** *Let $I \leq R$ be an ideal. The arithmetic rank of $I$, denoted $\operatorname{ara}(I)$, is the least number of elements of $R$ required to generate an ideal $J$ such that $\sqrt{J} = \sqrt{I}$.*

Note that if $b_1, \ldots, b_n \in R$ with $\sqrt{(b_1, \ldots, b_n)} = \sqrt{I}$, then for each $b_i$ there exists some $k_i$ such that $b_i^{k_i} \in I$. Since $\sqrt{I} = \sqrt{(b_1, \ldots, b_n)} \subseteq \sqrt{(b_1^{k_1}, \ldots, b_n^{k_n})} \subseteq \sqrt{I}$, we may equivalently define $\operatorname{ara}(I)$ as the minimum number of elements of $I$ whose radical ideal gives the radical of $I$.

**Corollary 2** (Minimal Generation up to Radical)**.** $H_I^i(M) = 0$ for all $i > \operatorname{ara}(I)$ and all $R$-modules $M$.

*Proof.* Let $J$ be the ideal generated by the $\operatorname{ara}(I)$ elements such that $\sqrt{J} = \sqrt{I}$. Proposition 5 implies $H_J^i(M) = 0$ for all $i > \operatorname{ara}(I)$. Since $\sqrt{J} = \sqrt{I}$, Proposition 4 yields $H_I^i(M) = 0$ for all $i > \operatorname{ara}(I)$. □

In particular Corollary 2 applies to a classic problem in algebraic geometry which asks whether it is possible for the ideal $I = (x, y) \cap (u, v)$ in $R = k[x, y, u, v]$, which is generated by the four elements $xu, xv, yu, yv$ and generated up to radical by $xu, yv, xv+yu$, to be generated by fewer than three elements up to radical. Appeals to properties of local cohomology such as Mayer-Vietoris reveal that $H_I^3(R) \neq 0$, and hence the answer is *no* [1].

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