Kunneth for Homology

Goal: understanding homology of product of spaces from their individual homologies.

Part 1: CW Complexes

$\boxed{\text{Thm}}$ If X & Y are CW complexes, then $X \times Y$ is a CW complex s.t. $C_*^{CW}(X\times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)$

Note: need to first define tensor product of chain complexes.

Given chain complexes $\mathcal{C}$ and $\mathcal{D}$, can define their tensor product $\mathcal{C}\otimes\mathcal{D} := \bigoplus_{n\in\mathbb{Z}} (\mathcal{C}\otimes\mathcal{D})_n$ where $(\mathcal{C}\otimes\mathcal{D})_n = \bigoplus_{p+q=n} \mathcal{C}_p \otimes \mathcal{D}_q$ with a boundary map given by

$$\bigoplus_{p+q=n} \mathcal{C}_p\otimes\mathcal{D}_q \xrightarrow{\partial_\otimes} \bigoplus_{p+q=n-1} \mathcal{C}_p\otimes\mathcal{D}_q$$

$$\partial_\otimes(a\otimes b) = \partial a\otimes b + (-1)^p a\otimes\partial b.$$

! Should verify that $\partial_\otimes$ is actually a boundary map, that is, $\partial_\otimes^2 = 0$. Works out b/c of signs!

$$\partial_\otimes(\partial_\otimes(a\otimes b)) = \partial_\otimes\big(\partial a\otimes b + (-1)^p a\otimes\partial b\big) = \cancel{\partial^2 a\otimes b} + (-1)^{p-1}\partial a\otimes\partial b + (-1)^p \partial a\otimes\partial b + \cancel{(-1)^{2p} a\otimes\partial^2 b}$$
$$= 0.$$

To prove the theorem: not hard to see that $X\times Y$ is a CW complex, with n-cells consisting of $e_\alpha^i \times e_\beta^{n-i}$ for e_α^i in X & e_β^{n-i} in Y. So n-skeleton is $(X\times Y)_n = \bigcup_{p+q=n} X^p \times Y^q$.

To see that $C_*^{CW}(X\times Y) \cong C_*^{CW}(X)\otimes C_*^{CW}(Y)$, define

$$C_*^{CW}(X)\times C_*^{CW}(Y) \xrightarrow{\times} C_*^{CW}(X\times Y)$$
$$\underbrace{e_\alpha^p \times e_\beta^q}_{\in C_p^{CW}(X)\times C_q^{CW}(Y)} \longmapsto \underbrace{e_\alpha^p\times e_\beta^q}_{\in C_{p+q}^{CW}(X\times Y)}$$

$\times$ is clearly bilinear, so induces a map

$$C_*^{CW}(X)\otimes C_*^{CW}(Y) \xrightarrow{\times} C_*^{CW}(X\times Y)$$
$$e_\alpha^p \otimes e_\beta^q \longmapsto e_\alpha^p \times e_\beta^q.$$

Injectivity & surjectivity are clear.

need to show that the cellular boundary map $C_n^{CW}(X\times Y) \xrightarrow{\partial} C_{n-1}^{CW}(X\times Y)$ satisfies the same Leibniz rule as $\partial_\otimes$, that is, $\partial(e_\alpha^p \times e_\beta^q) = \partial e_\alpha^p \times e_\beta^q + (-1)^p e_\alpha^p \times \partial e_\beta^q$ for product cell in $C_n^{CW}(X\times Y)$.

This is hard!! We know how each cell is attached to X & Y explicitly, but difficult to understand the attaching map of the product.

Part 2: Cross Product Map on Singular Chain Complexes

When X & Y are not CW complexes, we don't have an obvious map $C_*(X) \otimes C_*(Y) \xrightarrow{\times} C_*(X \times Y)$.
(We also don't have an isomorphism anymore, but we do have a chain homotopy equivalence.)
We should define a cross product map $\times$ (and then an inverse map $\theta: C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$) such that $\times \circ \theta \simeq \mathrm{id}_{C_*(X \times Y)}$ and $\theta \circ \times \simeq \mathrm{id}_{C_*(X) \otimes C_*(Y)}$ where the homotopy equivalences are <u>chain homotopy equivalences</u>, ie, there exists a sequence of maps $\{D_n : C_n(X \times Y) \longrightarrow C_{n+1}(X \times Y)\}$ such that $D_{n-1} \circ \partial + \partial \circ D_n = \mathrm{id} - \times \circ \theta$

(and a similar construction for $\theta \circ \times$).

$$\begin{tikzcd}
& C_n(X \times Y) \arrow[r, "\partial"] \arrow[dl, "D_n"'] \arrow[d, shift right, "\times \circ \theta"'] \arrow[d, shift left, "\mathrm{id}"] & C_{n-1}(X \times Y) \arrow[dl, "D_{n-1}"] \\
C_{n+1}(X \times Y) \arrow[r, "\partial"] & C_n(X \times Y) &
\end{tikzcd}$$

Since chain homotopy equivalent chain complexes have isomorphic homology, such a construction would induce

$$\star\star \quad H_n(X \times Y) \cong H_n(C_*(X) \otimes C_*(Y)) \quad \text{for all } n. \quad \star\star$$

and then our task would be to analyze the singular homology of $C_*(X) \otimes C_*(Y)$!
(Natural question: Is $H_n(C_*(X) \otimes C_*(Y)) \cong \bigoplus_{i=0}^{n} H_i(X) \otimes H_{n-i}(Y)$? We'll see later that in general, the answer is <u>no</u>.)

<u>Step 1</u>: Define $\times : C_*(X) \times C_*(Y) \longrightarrow C_*(X \times Y)$
such that:

(i) $\times$ is $\mathbb{Z}$-bilinear (hence induces linear map $C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ by universal property of tensor products.

(ii) boundary map on $C_*(X \times Y)$ commutes with that of tensor product, i.e.,

$$d(a \times b) = \partial a \times b + (-1)^{|a|} a \times \partial b$$

(iii) cross product is <u>natural</u>: if $f: X \to X'$ & $g: Y \to Y'$ are maps, $a \in C_p(X)$, $b \in C_q(Y)$, then

$$\underbrace{\underbrace{f_\#(a)}_{\in C_p(X')} \times \underbrace{g_\#(b)}_{\in C_q(Y')}}_{\in C_{p+q}(X' \times Y')} = \underbrace{(f \times g)_\# \underbrace{(a \times b)}_{\in C_{p+q}(X \times Y)}}_{\in C_{p+q}(X' \times Y')}$$

where $f \times g$ is the cartesian product of maps $X \times Y \longrightarrow X' \times Y'$.

(iv) normalization: given $x \in X$, $y \in Y$ (viewed as 0-singular simplices)

define $j_x : Y \to X \times Y$, $y \mapsto (x,y)$ $\quad$ $\iota_y : X \to X \times Y$, $x \mapsto (x,y)$.

if $b \in C_q(Y)$, define $\underbrace{c_x^0 \times b}_{\text{should lie in } C_q(X\times Y)} := (j_x)_\#(b)$. if $a \in C_p(X)$, define $\underbrace{a \times c_y^0}_{\text{should lie in } C_p(X\times Y)} := (\iota_y)_\#(a)$

Construction: We construct this map inductively on $p+q=n$.

When $p=0$ or $q=0$ we're done by (iv) normalization. So $(p,q)=(0,0)$ is done.

Suppose $p+q=1$ w/ $p=0$, $q=1$. Define $C_0(X) \times C_1(Y) \xrightarrow{\times} C_1(X\times Y)$

$$\sum a_i x_i \times b := \sum a_i (j_{x_i})_\#(b).$$

? Does the differential satisfy Leibniz? $d(x_i \times b)$ should intuitively be $x_i \times \partial b$, and indeed, $d(x_i \times b) = d((j_{x_i})_\#(b))$

$= (j_{x_i})_\# \, db$ $\quad$ since $(j_x)_\#$ is a chain map, commutes with boundary

$= x_i \times \partial b$

Similary story holds for $p=1$, $q=0$.

Now suppose a map satisfying (i)–(iv) is constructed for $p+q=n-1$.

We need to define $\sigma \times \tau \in C_{p+q}(X\times Y)$ for $\sigma \in C_p(X)$, $\tau \in C_q(X)$ with $p+q=n$.

Recall: $\sigma \in C_p(X)$ means σ is a linear combination of singular p-simplices on X, which are continuous maps from the standard p-simplex $\Delta^p \to X$. Without loss of generality we may assume σ and τ are themselves singular simplices (so $\sigma : \Delta^p \to X$, $\tau : \Delta^q \to Y$) and then extend the cross product defintion bilinear-ly.

! It's tempting here to define $\sigma \times \tau$ as the cartesian product $\sigma \times \tau : \Delta^p \times \Delta^q \to X \times Y$. But elements of $C_{p+q}(X\times Y)$ are linear combinations of maps $\Delta^{p+q} \to X \times Y$, and while $\Delta^p \times \Delta^q \underset{\text{homeo}}{\cong} \Delta^{p+q}$, there is no canonical identification of the two. Thus we need to appeal to another strategy.

Strategy: Use naturality. We have the p-singular simplex $\iota_p : \Delta^p \to \Delta^p \in C_p(\Delta^p)$ which is simply the identity.

We similarly have $\iota_q : \Delta^q \to \Delta^q$. Now $\sigma = \sigma_\#(\iota_p) \in C_p(X)$ and $\tau = \tau_\#(\iota_q)$ — this is pretty tautological!

as a map $C_p(\Delta^p) \to C_p(X)$

but naturality tells us that if we define $\iota_p \times \iota_q$ as an element of $C_{p+q}(\Delta^p \times \Delta^q)$, then $\sigma \times \tau := \underbrace{(\sigma \times \tau)}_{\substack{\text{cartesian product of} \\ \text{the maps } \sigma : \Delta^p \to X \\ \text{and } \tau : \Delta^q \to Y}}{}_\#(\iota_p \times \iota_q)$.

Thus suffices to define $\iota_p \times \iota_q$! The trick here is to notice that $\Delta^p \times \Delta^q$ is contractible, hence has trivial reduced homology groups (so all cycles are boundaries).

We outline two ways to define $\iota_p \times \iota_q$:

Method 1: Acyclic Models

If $\iota_p \times \iota_q$ were defined, we know what it's boundary should be: "$d(\iota_p \times \iota_q)$" $= d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q$.

Note that the right hand side is a well-defined element of $C_{n-1}(\Delta^p \times \Delta^q)$ by the inductive hypothesis.

We can actually check that the right hand side of "$d(\iota_p \times \iota_q)$" is a cycle: $d(d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q) = 0$ (see above computation for $\partial_{\otimes}^2 = 0$). Thus $d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q$ is a well-defined element of $H_{n-1}(\Delta^p \times \Delta^q) = 0$ for $n \geq 2$ (and we may assume $n \geq 2$ since we explicitly constructed the base cases of $n=0, n=1$).

So $d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q$ is actually the boundary of <u>some</u> element $\beta \in C_n(\Delta^p \times \Delta^q)$, and we can define $\iota_p \times \iota_q$ to be this β. Note that there is a non-canonical choice being made here, since in theory there could be several β for which $d\beta$ gives $d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q$.

Now $\sigma \times \tau := (\sigma \times \tau)_\#(\iota_p \times \iota_q)$ and we verify that the boundary map works as expected:

$$\begin{aligned}
d(\sigma \times \tau) &= d((\sigma \times \tau)_\#(\iota_p \times \iota_q)) \\
&= (\sigma \times \tau)_\#(d(\iota_p \times \iota_q)) && \text{(chain maps commute with boundaries)} \\
&= (\sigma \times \tau)_\#(d\iota_p \times \iota_q + (-1)^p \iota_p \times d\iota_q) \\
&= (\sigma \times \tau)_\#(d\iota_p \times \iota_q) + (-1)^p (\sigma \times \tau)_\#(\iota_p \times d\iota_q) && \text{(linearity)} \\
&= \sigma_\#(d\iota_p) \times \tau_\#(\iota_q) + (-1)^p(\sigma_\#(\iota_p) \times \tau_\#(d\iota_q)) && \text{(naturality in the case of } p+q=n-1 \text{, by inductive hypothesis)} \\
&= d\sigma_\#(\iota_p) \times \tau_\#(\iota_q) + (-1)^p \sigma_\#(\iota_p) \times d\tau_\#(\iota_q) && \text{(chain maps commute with boundaries)} \\
&= d\sigma \times \tau + (-1)^p \sigma \times d\tau. \ \square
\end{aligned}$$

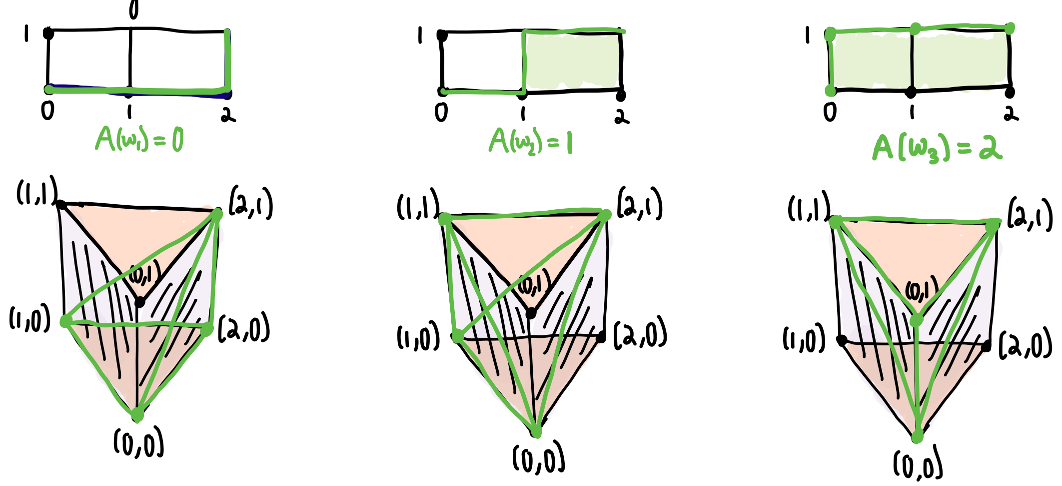
Method 2: Explicit Construction via Eilenberg-Zilber Chain

(due to Eilenberg & MacLane, description appears in 1940 paper by Eilenberg & Moore).

Goal is to triangulate the prism: find simplices Δ^{p+q} inside of $\Delta^p \times \Delta^q$, assign to each simplex a coefficient as follows:

- simplices are indexed by order-preserving injections $\omega : [p+q] \longrightarrow [p] \times [q]$
  ↳ since ω is injective, we know $\omega(0) = (0,0)$ and $\omega(p+q) = (p,q)$.
- each ω can be depicted as a right-up staircase in $[0,p] \times [0,q]$, and we define $A(\omega)$ to be the area under this staircase. Then define $\boxed{\iota_p \times \iota_q := \sum_\omega (-1)^{A(\omega)} \omega}$

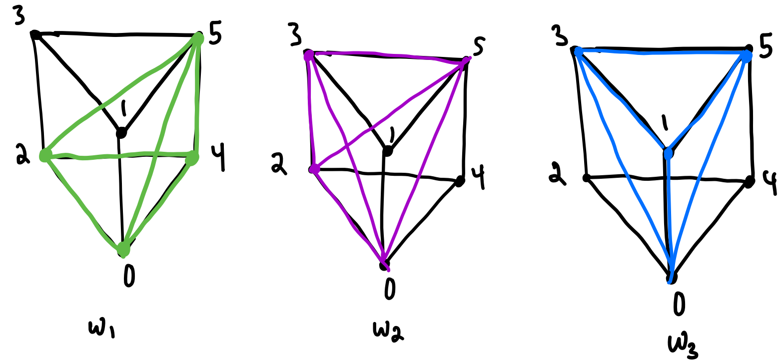
$\boxed{\text{EX}}$ $p=2, q=1$. Looking for Δ^3's in $\Delta^2 \times \Delta^1$.



Then $\iota_2 \times \iota_1$ is defined as $\sum_{i=1}^{3} (-1)^{A(\omega_i)} \omega_i = \omega_1 - \omega_2 + \omega_3$ where each ω_i is the 3-simplex depicted above, respectively.

What is the boundary of $\iota_2 \times \iota_1$? Assign lexicographic order to vertices (total order).

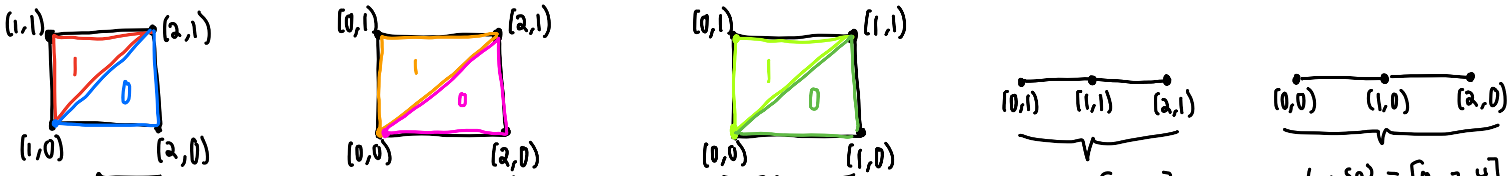
Relabel: $(0,0) \longmapsto 0$ $(0,1) \longmapsto 1$ $(1,0) \longmapsto 2$ $(1,1) \longmapsto 3$ $(2,0) \longmapsto 4$ $(2,1) \longmapsto 5$



$$\partial(w_1) = [2\,4\,5] - [0\,4\,5] + \cancel{[0\,2\,5]} - [0\,2\,4]$$
$$\partial(w_2) = [2\,3\,5] - \cancel{[0\,3\,5]} + \cancel{[0\,2\,5]} - [0\,2\,3]$$
$$\partial(w_3) = [1\,3\,5] - \cancel{[0\,3\,5]} + [0\,1\,5] - [0\,1\,3]$$

$$\Rightarrow \partial(c_2 \times c_1) = \partial w_1 - \partial w_2 + \partial w_3$$
$$= [2\,4\,5] - [0\,4\,5] - [0\,2\,4] - [2\,3\,5] + [0\,2\,3] + [1\,3\,5] + [0\,1\,5] - [0\,1\,3]$$

What is $dc_2 \times c_1 + c_2 \times dc_1$? $\quad dc_2 \times c_1 = [1\,2] \times c_1 - [0\,2] \times c_1 + [0\,1] \times c_1 + c_2 \times \{1\} - c_2 \times \{0\}$



$[1,2] \times c_1 = [2\,4\,5] - [2\,3\,5]$ $\quad [0,2] \times c_1 = [0\,4\,5] - [0\,1\,5]$ $\quad [0,1] \times c_1 = [0\,2\,3] - [0\,1\,3]$ $\quad c_2 \times \{1\} = [1\,3\,5]$ $\quad c_2 \times \{0\} = [0\,2\,4]$

in vertex labels above

So $dc_2 \times c_1 + c_2 \times dc_1 = [2\,4\,5] - [2\,3\,5] - [0\,4\,5] + [0\,1\,5] + [0\,2\,3] - [0\,1\,3] + [1\,3\,5] - [0\,2\,4] = d(c_2 \times c_1)$!

Part 3: Dual Product

We need to define a map $\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ which is a chain homotopy inverse of $\times$.

Strategy: Again using acyclic models — define θ on "model simplices" $d_n : \Delta^n \to \Delta^n \times \Delta^n$ given by diagonal inclusion. $v \mapsto (v, v)$.

Recall in proof of $\times$ map, it was useful that $\tilde{H}_n(\Delta^p \times \Delta^q) = 0$ for all n.
Here it will be useful to have that $\tilde{H}_k(C_*(\Delta^n) \otimes C_*(\Delta^n)) = 0$ for all k.
This is not as trivial! We need the following lemma:

Lemma: (acyclic models) If X and Y are contractible spaces, then $H_n(C_*(X) \otimes C_*(Y)) \cong \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$.

proof: We wish to construct a chain homotopy between the chain maps $id \otimes id$ and $\varepsilon \otimes \varepsilon$, where ε is the augmentation map on $C_*(X)$ which is 0 in deg > 0 and $C_0(X) \xrightarrow{\varepsilon} C_0(X)$, $\sum n_i \sigma_i \mapsto \sum n_i x_0$

If we can show this, then because homotopic chain maps yield same maps on homology $\Rightarrow H_*(C_*(X) \otimes C_*(Y))$ may be calculated via $\varepsilon \otimes \varepsilon$, which is trivial except in degree 0.

Since X is contractible, the identity map is null homotopic; call this null homotopy F, so $F_0 = id$, $F_1 = \{x_0\}$.
We claim that the identity chain map induced by id is chain homotopic to the augmentation chain map ε given above.

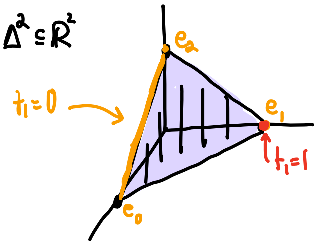
Recall: $\Delta^n = \{ \sum_{i=0}^{n} t_i e_i \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \}$, where $\vec{e}_i$ is (i+1)'st basis vector of $\mathbb{R}^{n+1}$.

We can thus view an n-simplex $\sigma: \Delta^n \to X$ as a map on the Barycentric coordinates $(t_0, \dots, t_n)$.

The faces of Δ^n are given by $t_i = 0$ for $i = 0, \dots, n$

— vertices ———————— $t_i = 1$ ————.

$\boxed{\text{EX}}$ $\Delta^2 \subseteq \mathbb{R}^2$



Given an n-simplex $\sigma: \Delta^n \to X$, define $D_n(\sigma)$ to be the (n+1)-simplex $D_n(\sigma)(t_0, \dots, t_{n+1}) = F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right)\right)$

(Check: $\sum_{i=0}^{n+1} t_i = 1 \Rightarrow \sum_{i=1}^{n+1} \frac{t_i}{1-t_0} = \frac{1}{1-t_0}(1-t_0) = 1$.)

Let's look at an example.

Suppose $\sigma: \Delta^2 \to X$. Then $D_1(\partial\sigma) = D_1\left(\sum_{i=0}^{2} (-1)^i \sigma^i\right) = \sum_{i=0}^{2} (-1)^i D_1(\sigma^i)$.

Recall that σ^i is the restriction of σ to its i^{th} face, which is opposite of vertex e_i. Hence $\sigma^0 = \sigma|_{[e_1\, e_2]} \in C_1(X)$.

Then $D_1(\sigma^0)$ is a map $\Delta^2 \to X$ which takes in the Barycentric coordinates $t_0, \dots, t_2$ and spits out $F_{t_0}\left(\sigma^0\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right)$.

We see that this corresponds to the point $\left(0, \frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)$ in the original 2-simplex Δ^2.

Similarly: $D_1(\sigma^1)(t_0, t_1, t_2) = F_{t_0}\left(\sigma^1\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right) = F_{t_0}\left(\sigma|_{[e_0\, e_2]}\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right) = F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, 0, \frac{t_2}{1-t_0}\right)\right)$

$D_1(\sigma^2)(t_0, t_1, t_2) = F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, 0\right)\right)$

Thus: $D_1(\partial\sigma)(t_0, t_1, t_2) = F_{t_0}\left(\sigma^0\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right) - F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, 0, \frac{t_2}{1-t_0}\right)\right) + F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, 0\right)\right)$

On the other hand, $\partial D_2(\sigma)(t_0, t_1, t_2) = \sum_{i=0}^{3} (-1)^i D_2(\sigma)^i (t_0, t_1, t_2)$

$$= D_2(\sigma)\big|_{[e_1\, e_2\, e_3]}(t_0, t_1, t_2) - D_2(\sigma)\big|_{[e_0\, e_2\, e_3]}(t_0, t_1, t_2) + D_2(\sigma)\big|_{[e_0\, e_1\, e_3]}(t_0, t_1, t_2) - D_2(\sigma)\big|_{[e_0\, e_1\, e_2]}(t_0, t_1, t_2)$$

$$= D_2(\sigma)(0, t_0, t_1, t_2) - D_2(\sigma)(t_0, 0, t_1, t_2) + D_2(\sigma)(t_0, t_1, 0, t_2) - D_2(\sigma)(t_0, t_1, t_2, 0)$$

$$= F_0(\sigma(t_0, t_1, t_2)) - F_{t_0}\left(\sigma\left(0, \frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right) + F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, 0, \frac{t_2}{1-t_0}\right)\right) - F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, 0\right)\right)$$

$$= \sigma(t_0, t_1, t_2) - F_{t_0}\left(\sigma\left(0, \frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}\right)\right) + F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, 0, \frac{t_2}{1-t_0}\right)\right) - F_{t_0}\left(\sigma\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, 0\right)\right)$$

since $F_0 = \mathrm{id}$.

We see that $D_1(\partial\sigma) + \partial(D_2(\sigma)) = \sigma$, so $D_1 \circ \partial + \partial \circ D_2 = \mathrm{id}$.

More generally: $D_{n-1} \circ \partial + \partial \circ D_n = \mathrm{id}$ for $n \geq 1$.

When $n = 0$, one can check that $\partial D_0(\sigma) + x_0 = \sigma$, so $\partial D_0(\sigma) = \sigma - x_0 = \sigma - \varepsilon(\sigma)$

This shows that $D \circ \partial + \partial \circ D = \mathrm{id} - \varepsilon$, so id & ε are chain homotopic. □

We have such a chain homotopy for Y too. We wish to combine these into a chain homotopy of $\mathrm{id} \otimes \mathrm{id}$ and $\varepsilon \otimes \varepsilon$.

Define $Q: (C_*(X)\otimes C_*(Y))_n \longrightarrow (C_*(X)\otimes C_*(Y))_{n+1}$
$$Q(a\otimes b) = (D\otimes\varepsilon)(a\otimes b) + (-1)^{|a|}(id\otimes D)(a\otimes b).$$

Then
$$\partial_\otimes Q + Q\partial_\otimes (a\otimes b) = \partial_\otimes\left(D\otimes\varepsilon(a\otimes b) + (-1)^{|a|}(id\otimes D)(a\otimes b)\right) + Q\left(\partial a\otimes b + (-1)^{|a|} a\otimes\partial b\right)$$
$$= \partial_\otimes\left(D(a)\otimes\varepsilon(b) + (-1)^{|a|} a\otimes D(b)\right) + Q(\partial a\otimes b) + (-1)^{|a|} Q(a\otimes\partial b)$$
$$= (\partial D + D\partial)\otimes\varepsilon(a\otimes b) + id\otimes(\partial D + D\partial)(a\otimes b)$$
$$= ((id-\varepsilon)\otimes\varepsilon + id\otimes(id-\varepsilon))(a\otimes b)$$

$\Rightarrow \partial_\otimes Q + Q\partial_\otimes = id\otimes id - \varepsilon\otimes\varepsilon$. □

We are now set to actually define our dual chain map!

Step 2: Define a map $\theta: C_*(X\times Y)\longrightarrow C_*(X)\otimes C_*(Y)$ such that

(i) on 0-chains θ is obvious: $\theta(x,y) = x\otimes y$ ← $\Delta^0\to Y$
← $\Delta^0\to X$
↑ viewed as a 0-simplex $\Delta^0\to X\times Y$

(ii) $\partial_\otimes\circ\theta = \theta\circ\partial$

(iii) naturality: if $f: X\longrightarrow X'$, $g: Y\longrightarrow Y'$ are continuous maps, then $\theta\circ(\underbrace{f\times g}_{\text{cartesian product}})_\# = (f_\#\otimes g_\#)\circ\theta$

proof: induction & use of acyclic models:

Case $n=0$ is done by map in (i).

Now suppose $\theta: C_k(X\times Y)\longrightarrow (C_*(X)\otimes C_*(Y))_k$ has been defined for chains of deg $k\le n-1$.

Consider $d_n: \Delta^n\longrightarrow\Delta^n\times\Delta^n$ diagonal inclusion (so $d_n\in C_n(\Delta^n\times\Delta^n)$).

Again we compute the formal boundary "$\partial_\otimes\theta(d_n)$" $= \theta(\underbrace{\partial d_n}_{\in C_{n-1}(\Delta^n\times\Delta^n)})$ by property (ii), and θ is defined for $k=n-1$.

Now the right hand side is a cycle since $\partial_\otimes(\theta(\partial d_n)) = \theta(\partial\partial d_n) = \theta(0) = 0$. By the lemma, we know that $H_n(C_*(\Delta^n)\otimes C_*(\Delta^n)) = 0$ for $n\ge 1$ since Δ^n is contractible. So $\theta(\partial d_n)\in H_{n-1}(\Delta^n\times\Delta^n) = 0$ for $n\ge 2$ (what happens for $n=1$?).

Thus $\theta(\partial d_n)$ is the boundary of some $\beta\in(C_*(\Delta^n)\otimes C_*(\Delta^n))_n$, and we define $\theta(d_n) = \beta$.

More generally: if $\sigma: \Delta^n\longrightarrow X\times Y$ is an n-simplex, we have $\pi_X\circ\sigma: \Delta^n\longrightarrow X$ and $\pi_Y\circ\sigma: \Delta^n\longrightarrow Y$, so $(\pi_X\circ\sigma\times\pi_Y\circ\sigma)_\#: C_*(\Delta^n\times\Delta^n)\longrightarrow C_*(X\times Y)$.

Then as an n-simplex, σ is given by $(\pi_X\circ\sigma\times\pi_Y\circ\sigma)_\#(d_n)$, so by naturality we have $\theta(\sigma) = \theta((\pi_X\circ\sigma\times\pi_Y\circ\sigma)_\#(d_n))$
↑ $\in C_n(\Delta^n\times\Delta^n)$
$$= (\pi_X\circ\sigma)_\#\otimes(\pi_Y\circ\sigma)_\#(\theta(d_n)).$$

Similar computation as that for $\times$ shows that $\partial_\otimes\theta(\sigma) = \theta(\partial\sigma)$. □

Lemma: $\times$ and θ are chain-homotopy inverses.

proof: (skipped).

Part 4: Künneth Theorem for Homology

Note: $\times$ takes cycles ⊗ cycles to cycles and cycles ⊗ boundaries to boundaries:

- if $\partial a = 0$ & $\partial b = 0$, then $\partial(a\times b) = \partial a\times b + (-1)^{|a|} a\otimes\partial b = 0$.
- if $\partial a = 0$ & $b = \partial c$, then $a\times b = \partial a\times c + a\times\partial c = (-1)^{|a|}\partial(a\times c)$

Thus $\times$ induces a well-defined map on homology:

$\times: H_p(X) \otimes H_q(Y) \longrightarrow H_{p+q}(X \times Y)$

$[\sigma] \otimes [\tau] \longmapsto [\sigma \times \tau]$ (independent of choices: for instance, if σ' differs from σ by a boundary $\partial\gamma$, then $\sigma \times \tau - \sigma' \times \tau = \partial\gamma \times \tau = \partial(\gamma \times \tau)$ since $\partial\tau = 0$.)

Now we may apply this component-wise? to $\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y)$. What can we say about this map?

$\boxed{\text{Thm}}$ (Kunneth - Algebraic): Let R be a PID, R-modules C_i are free, then for each n $\exists$ natural SES

$$0 \longrightarrow \bigoplus_i H_i(C) \otimes_R H_{n-i}(C') \longrightarrow H_n(C \otimes_R C') \longrightarrow \bigoplus_i \mathrm{Tor}_R(H_i(C), H_{n-i-1}(C')) \longrightarrow 0$$

which splits (although not naturally).

proof: Special case first where boundary maps of C are all 0. Then $H_i(C) \cong C_i$. Also then $C_p \otimes C'_q \xrightarrow{\partial} C_p \otimes C'_{q-1}$ since $\partial(c \otimes c') = (-1)^p c \otimes \partial c'$.

Hence $C \otimes C'$ is the direct sum of complexes $C_i \otimes_R C'$.

Now $\bigoplus_i C_i \otimes C'_{n-i+1} \xrightarrow{id \otimes \partial} \bigoplus_i C_i \otimes C'_{n-i} \xrightarrow{id \otimes \partial} \bigoplus_i C_i \otimes C'_{n-i-1}$ and for a fixed i, get $C_i \otimes C'_{n-i+1} \xrightarrow{id \otimes \partial} C_i \otimes C'_{n-i}$.

Thus $H_n(C_i \otimes_R C') \cong C_i \otimes H_{n-i}(C') = H_i(C) \otimes H_{n-i}(C')$.

So $H_n(C \otimes C') \cong H_n(\bigoplus_i C_i \otimes C') \cong \bigoplus_i H_i(C) \otimes H_{n-i}(C')$ and since free modules are flat, $\mathrm{Tor}(H_i(C), H_{n-i-1}(C')) = 0$ $\forall i$.

General case: Let $Z_i \subset C_i$ & $B_i \subset C_i$ denote kernel & image of boundary hom. for C.

Obtain subchain complexes

$$\to Z_n \to Z_{n-1} \to \cdots$$

$$\to B_n \to B_{n-1} \to \cdots$$ with trivial boundary maps (boundary of cycle is 0, boundary of boundary is 0).

Get a SES of chain complexes $0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0$ which splits since B_{i-1} is a submodule of a free R-module over a PID.

Because this splits, tensor with C' and remains split exact:

$$0 \to Z_* \otimes C' \longrightarrow C \otimes C' \longrightarrow B_{*-1} \otimes C' \longrightarrow 0$$

and from a SES of chain complexes we know we get a long exact sequence in homology:

$$\to H_{n+1}(B_{*-1} \otimes C') \longrightarrow H_n(Z_* \otimes C') \longrightarrow H_n(C \otimes C') \longrightarrow H_n(B_{*-1} \otimes C') \longrightarrow H_{n-1}(Z_* \otimes C') \to$$

The connecting homomorphism is given by $\iota \otimes \mathrm{id}$ where ι is the inclusion of $B_i \hookrightarrow Z_i$.

By the special case, since Z_* & B_* are trivial complexes, we know $H_n(Z_* \otimes C') \cong \bigoplus_i Z_i \otimes H_{n-i}(C')$

$$H_n(B_{*-1} \otimes C') \cong \bigoplus_i B_i \otimes H_{n-i-1}(C').$$

So our LES of homology becomes

$$\xrightarrow{i_n \otimes \mathrm{id}} \bigoplus_i Z_i \otimes H_{n-i}(C') \xrightarrow{j} H_n(C \otimes C') \xrightarrow{\ell} \bigoplus_i B_i \otimes H_{n-i-1}(C') \xrightarrow{i_{n-1}}$$

which yields the SES

$$0 \to \underset{\operatorname{im} j}{\overset{\|}{\operatorname{coker} i_n}} \hookrightarrow H_n(C \otimes C') \twoheadrightarrow \underset{\operatorname{im} \ell}{\overset{\|}{\ker i_{n-1}}} \to 0.$$

Now $\operatorname{coker} i_n = \bigoplus_i Z_i \otimes_R H_{n-i}(C') / \operatorname{im} i_n \cong \bigoplus_i H_i(C) \otimes H_{n-i}(C')$ since $0 \to B_{i+1} \to Z_i \to H_i(C) \to 0$ exact

$\Rightarrow B_{i+1} \otimes H_{n-i}(C') \xrightarrow{i_n} Z_i \otimes H_{n-i}(C') \to H_i(C) \otimes H_{n-i}(C') \to 0$ exact

It thus suffices to compute $\ker i_{n-1}$.

By definition, Tor completes an exact sequence

$$B_{i+1} \otimes H_{n-i}(C') \xrightarrow{i_n} Z_i \otimes H_{n-i}(C') \longrightarrow H_i(C) \otimes H_{n-i}(C') \to 0$$

to an exact sequence

$$0 \to \operatorname{Tor}_R(H_i(C), H_{n-i}(C')) \longrightarrow B_{i+1} \otimes H_{n-i}(C') \xrightarrow{i_n} Z_i \otimes H_{n-i}(C') \longrightarrow H_i(C) \otimes H_{n-i}(C') \to 0$$

So summing over i, $\ker i_{n-1} = \bigoplus_i \operatorname{Tor}_R(H_i(C), H_{n-i-1}(C'))$. $\square$

Consequence (Topological Künneth)

We showed $H_*(X \times Y) \cong H_*(C_*(X) \otimes C_*(Y))$.

Thus over PIDs R, we have a natural SES which splits (non-naturally):

$$0 \to \bigoplus_i H_i(X;R) \otimes H_{n-i}(Y;R) \longrightarrow H_n(X \times Y; R) \longrightarrow \bigoplus_i \operatorname{Tor}_R(H_i(X;R), H_{n-i-1}(Y;R)) \to 0.$$

Corollary 1: If R is a field, all R-modules are free (every vector space has a basis) so Tor vanishes, obtain "desired" isomorphism.

Corollary 2: If C' is the chain complex $\to 0 \to \cdots \to 0 \to \underset{\text{deg. } 0}{G}$, then algebraic Künneth yields, for each n,

$$0 \to H_n(X) \otimes G \to H_n(C \otimes G) \to \mathrm{Tor}_R(H_{n-1}(C), G) \to 0 \quad \text{(split exact)}.$$

yielding the universal coefficient theorem for homology.