From K_0 to higher algebraic K-theory

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Abstract

Originating from the German word "klasse" (class), both topological and algebraic K-theory operates under the philosophy that studying certain isomorphism classes of objects over a space/ scheme/ ring/ category is a good way to study that space (/scheme/ ring / category). In the case of topological K-theory, we'll consider isomorphism classes of vector bundles over a space X, whereas in algebraic K-theory we'll look at finitely generated projective modules over R. The Serre-Swan theorem will allow us to reconcile these stories on the level of K_0. We'll finish the talk with an algebraic description of $K_1(R)$ and $K_2(R)$, as well as Quillen's "+" construction for defining higher algebraic K-groups.

1 Introduction

Philosophy of K-theory: "The universal invariant". In algebraic topology we have functors from spaces to groups which allow us to distinguish between spaces. Similar idea here: often easier to compute some topological properties from the mapped rings than from the original spaces/ schemes/ categories.
Two main components to algebraic K-theory:

1. Classical: the Grothendieck group K_0 of a category (algebra).

2. Higher algebraic K-theory (topological/ homological).

Slogan: Algebraic K-theory deals with *linear algebra over general rings* R instead of over fields. Associate to R a sequence of abelian groups $K_i(R)$ whose behavior resembles a "homology theory," e.g. we have long exact sequences of pairs and functoriality.

Algebraic K-theory has wide-reaching connections to many fields. For example:

1. The class group of a number field K (measures failure of unique factorization of ideals in the ring of integers O_K) is approximately $K_0(O_K)$ (it's actually $\tilde{K}_0(O_K)$).

2. "Whitehead torsion" in topology (measures obstructions to $f : X \to Y$ a homotopy equivalence between CW complexes being a *simple* homotopy equivalence) is essentially an element in $K_1(\mathbb{Z}\pi_1(Y))$.

3. Higher K-groups of fields and rings of integers are related to special values of L-functions.

2 K_0

2.1 K_0 of a top space

K-theory traces its origins to *topological* K-theory.
"Topological K-theory is the idea that the set of bundles that a space admits is a good invariant of the space."

Definition 2.1 (VB(X))**.** Let X be a paracompact space (every open cover has a locally finite refinement– i.e. can find a refinement (more open sets, containment condition) such that each point in X has a neighborhood U_x intersecting only finitely many guys in the refinement. Examples: compact spaces, Euclidean space…). The sets $\mathrm{VB}_{\mathbb{R}}(X)$ and $\mathrm{VB}_{\mathbb{C}}(X)$ are isomorphism classes of real/ complex vector bundles over X.

These form an abelian monoid under Whitney sum (pullback bundle under diagonal map $\delta : X \to X \times X$, fiberwise direct sum of fibers. i.e., we have $E \times F \to X \times X$ the product of the bundles, then take the direct sum $E \oplus F := \delta^*(E \times F)$). Under tensor product, these form a commutative semiring. So we can group complete! (See next section.) Form $\mathrm{KO}(X), \mathrm{KU}(X)$, identity is $1 = [T^1]$ (trivial bundle).

Higher topological K groups are defined by taking suspensions:

Definition 2.2. $K^{-n}(X) := K(\Sigma^n X)$.

(Negative indices indicate that coboundary maps increase dimension.)

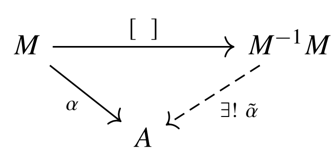
2.2 Group completion of a monoid

Definition 2.3 (Abelian monoid)**.** An abelian monoid is a set M together with an associative, commutative group operation $+$ and "additive identity" 0. (Group without inverses.)
A monoid map $f : M \to N$ is a *set map* such that $f(0) = 0$ and $f(m + m') = f(m) + f(m')$.

Example 2.4. $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

Example 2.5. If A is an abelian group, any additively closed subset of A containing 0 is an abelian monoid.

Definition 2.6 (Group completion)**.** Given an abelian monoid M, the group completion is an abelian group $M^{-1}M$ (formally "adding in inverses") together with a monoid map $[] : M \to M^{-1}M$, universal in the sense that, for every abelian group A and every monoid map $\alpha : M \to A$, there exists a unique abelian group homomorphism $\tilde{\alpha} : M^{-1}M \to A$ such that $\tilde{\alpha}([m]) = \alpha(m)$ for all $m \in M$.



Example 2.7. The group completion of $\mathbb{N}_0$ is $\mathbb{Z}$.

Prop 2.8. *Every abelian monoid has a group completion.*

Proof. Take the free abelian group $F(M)$ on symbols $[m]$ for $m \in M$, then factor out by the subgroup $R(M)$ generated by the relations $[m + n] - [m] - [n]$. □

Definition 2.9 ($K_0(X)$)**.** The Grothendieck group $K_0(X)$ for X a paracompact top space is the group completion of $\mathrm{VB}(X)$.

Example 2.10 ($X = *$)**.** If $*$ is a 1-point space, $K(*) = \mathbb{Z}$. Why? The iso classes of real vector bundles over the point are all the trivial bundles $\mathbb{R}^k \times \{*\}$. Determined by dimension. So $\mathrm{VB}(X) \cong \mathbb{N}_0$, and $K_0(X) = \mathbb{Z}$.

Lemma 2.11 (Contravariance of K)**.** *The functor $K(X)$ is contravariant in X: if $f : X \to Y$ is continuous, the induced bundles construction $E \to f^*E$ yields a function $f^* : \mathrm{VB}(Y) \to \mathrm{VB}(X)$.*

Corollary 2.12. *The universal map $X \to *$ induces a ring hom from $\mathbb{Z} = K(*)$ into $K(X)$. Sends $n > 0$ to the class of the trivial bundle $T^n = \mathbb{R}^n \times X$ over X. If $X \neq \varnothing$, then any point of X yields a map $* \to X$ splitting the universal map $X \to *$. So functoriality implies we get a map splitting $\mathbb{Z} \to K(X)$. So $\mathbb{Z}$ is a direct summand of $K(X)$ when $X \neq \varnothing$!*

Remark 2.13. By universality, if $M \to N$ is a monoid map, the map $M \to N \to N^{-1}N$ extends uniquely to a group homomorphism $M^{-1}M \to N^{-1}N$, so group completion is a *functor* Abelian Monoids $\to$ Ab. (Turns out this functor is left adjoint to the forgetful functor, so $\mathrm{Hom}_{\mathrm{Ab\,Mon}}(M, A) \cong \mathrm{Hom}_{\mathrm{Ab}}(M^{-1}M, A)$.)

Prop 2.14 (Characterization of the group completion)**.**

(a) Elements of $M^{-1}M$ are of the form $[m] - [n]$ for $m, n \in M$.

(b) If $m, n \in M$, then $[m] = [n]$ in $M^{-1}M \iff m + p = n + p$ for some $p \in M$.

(c) The monoid map $M \times M \to M^{-1}M$ sending $(m, n) \mapsto [m] - [n]$ is surjective.

(d) ($\implies$ $M^{-1}M$ is the quotient of $M \times M$ by $(m, n) \sim (m + p, n + p)$.)

Question: Does M inject into $M^{-1}M$ via $m \mapsto [m]$? Recall: $[m] = [n]$ in $M^{-1}M$ if and only if there exists $p \in M$ such that $m + p = n + p$. So if we can *cancel* p, then the answer is yes. If we can always cancel p we call M a cancellation monoid. ($\mathbb{N}_0$)

Definition 2.15 (Semiring)**.** An abelian monoid which also has an associative product which distributes over $+$, and a 2-sided multiplicative identity 1 (so a ring (not nec. commutative) but without subtraction!

Remark 2.16. The group completion of a semiring is a ring.

Definition 2.17. Let X be a topological space. The set $[X, \mathbb{N}]$ is the set of continuous maps $X \to \mathbb{N}$. This is a semiring under pointwise addition and multiplication. The group completion is $[X, \mathbb{Z}]$.

Example 2.18 (Representation ring)**.** Example from rep theory: Let G be a finite group, and let $\mathrm{Rep}_{\mathbb{C}}(G)$ be the set of finite-dimensional reps $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ up to isomorphism. This is an abelian monoid under $\oplus$. Maschke's Theorem $\implies$ $\mathbb{C}G$ (group algebra) is semisimple and $\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathbb{N}^r$, with $r = \#$ conjugacy classes of G (have a bijection between irreps and conjugacy classes).

So the group completion $R(G)$ of $\mathrm{Rep}_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{Z}^r$ as an abelian group. We also have a semiring structure on $\mathrm{Rep}_{\mathbb{C}}(G)$ via tensor product. So $R(G)$ is a commutative ring, the "representation ring" of G.

2.3 K_0 of a ring

Let R be a ring.

Definition 2.19 ($P(R)$)**.** Let $P(R)$ be the set of isomorphism classes of f.g. projective R-mods, together with $\oplus$ and 0. This forms an abelian monoid.

Definition 2.20 (Grothendieck group $K_0(R)$)**.** $K_0(R)$ is the group completion $P^{-1}P$ of $P(R)$. If R is commutative, $K_0(R)$ is a commutative ring with $1 = [R]$, since $P(R)$ is a commutative semiring with product $\otimes_R$. (We know $P \otimes_R Q \cong Q \otimes_R P$ and $P \otimes_R R \cong P$. If P, G are f.g. projective modules, so is $P \otimes_R Q$.)

Example 2.21 (Grothendieck group of fields, local rings, and PIDs)**.** Let k be a field (or division ring= field without commutativity).

What's $P(k)$? F.g. projective k-modules are just finite-dim. vector spaces, and isomorphism classes of k-vector spaces are determined by their dimension, so $P(k) \cong \mathbb{N}_0$, and $K_0(k) = \mathbb{Z}$.

Same argument shows that $K_0(R) = \mathbb{Z}$ for local rings R (f.g. projectives are free, so determined up to iso by rank). Same argument shows that $K_0(R) = \mathbb{Z}$ for R a PID (f.g. projectives over PIDs are free of finite rank, determined by rank).

Remark 2.22. Want to restrict to f.g. projectives because of the Eilenberg swindle: if R^∞ (free R-mod on countably infinite basis) is to be included, then $P \oplus R^\infty \cong R^\infty$ for P f.g. would yield $[P] = 0$ for all f.g. projective R-mods P, so $K_0(R) = 0$. Boring!!

Question: How to reconcile $K_0(R)$ with $K_0(X)$?

Theorem 2.23 (Serre-Swan Theorem)**.** *Slogan: "projective modules over commutative rings are like vector bundles on compact spaces." Let X be a compact Hausdorff space, and $C(X)$ the ring of continuous real (complex-)-valued functions on X. The category of real (complex) vector bundles on X is equivalent to the category of finitely generated projective modules over $C(X)$.*

The actual correspondence: Have a functor Γ

$$\Gamma : \mathrm{VB}_{\mathbb{C}}(X) \to \mathrm{ProjMod}(C(X))$$
$$E \mapsto \Gamma(X, E)$$

where $\Gamma(X, E)$ is a $C(X)$-module of sections. Swan's theorem says this functor is an equivalence of categories. ($\Gamma(X, E)$ is the space of global sections $s : X \to E$.)

2.4 Brief detour: connection with the Picard Group

Let R be a commutative ring.

Definition 2.24 (Rank)**.** The rank of a f.g. R-module M at a prime $\mathfrak{p} \leqslant R$ is $\operatorname{rk}_{\mathfrak{p}} M := \dim_{k(\mathfrak{p})}(M \otimes_R k(\mathfrak{p}))$ (where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$). Since $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong k(\mathfrak{p})^{\operatorname{rk}_{\mathfrak{p}}(M)}$, $\operatorname{rk}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_{\mathfrak{p}}$.

Remark 2.25. If P is a fg *projective* R-mod, then $\operatorname{rk}(P) : \mathfrak{p} \mapsto \operatorname{rk}_{\mathfrak{p}}(P)$ is a continuous function from the topological space $\operatorname{Spec}(R)$ (Zariski topology) to the discrete top space $\mathbb{N} \subseteq \mathbb{Z}$. (Why? Turns out: $P_{\mathfrak{p}} \cong (R_{\mathfrak{p}})^n$ for some $n \geqslant 0$ and there exists some $s \in R \backslash \mathfrak{p}$ such that $P_{\mathfrak{p}'} \cong (R_{\mathfrak{p}'})^n$ for all $\mathfrak{p}'$ not containing s (so the preimage of rank n is a union of $D(s)$).)

Definition 2.26 (Constant ranks)**.** Say that P has constant rank if $n = \operatorname{rk}_{\mathfrak{p}}(P)$ is independent of $\mathfrak{p}$.

Example 2.27. If $\operatorname{Spec} R$ is *topologically connected* (for example, if R is an integral domain), then every continuous map $\operatorname{Spec} R \to \mathbb{N}$ is constant, so every fg projective R-mod has constant rank.

Definition 2.28 (Algebraic line bundle)**.** An algebraic line bundle L over a comm ring R is a fg projective R-mod of constant rank 1.

Turns out: tensor product of line bundles is a line bundle: $(L \otimes_R M)_{\mathfrak{p}} \cong L_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ has rank 1 (rank multiplies over tensor products).

Definition 2.29 (Picard group)**.** $\operatorname{Pic}(R)$ is the set of isomorphism classes of algebraic line bundles over R. The tensor product endows $\operatorname{Pic}(R)$ with the structure of an abelian group, $[R]$ is the identity, and inverses are given by dual modules $\mathrm{Hom}_R(P,R)$: has rank 1 when P has rank 1, and f.g. / projective because P is. The evaluation map

$$\begin{aligned} P \otimes_R \check{P} &\xrightarrow{\text{eval}} R \\ p \otimes f &\mapsto f(p) \end{aligned}$$

is an isomorphism since being an isomorphism is a *local property*: If $\mathfrak{p}$ is a prime, then $(L \otimes_R \check{L})_{\mathfrak{p}} \cong L_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \check{L}_{\mathfrak{p}} \xrightarrow{\text{eval}} R_{\mathfrak{p}}$ is an isomorphism since $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ being rank 1?

2.4.1 Determinant line bundle

Definition 2.30 (det P)**.** Let $\det(P) = \bigwedge^n P$ where P is a projective module of constant rank n. ($\bigwedge^n P = P \otimes \cdots \otimes P / \langle m_1 \otimes \cdots \otimes m_n : m_i = m_j \text{ for some } i \neq j \rangle$.) This is a line bundle since it's projective, finitely generated, and of constant rank 1. ($\bigwedge^k P$ has constant rank $\binom{n}{k}$.)

Prop 2.31. $\det : K_0(R) \twoheadrightarrow P_0(R)$ *group homomorphism. (Suffices to show, by universal property of* K_0, $\det(P \otimes_R Q) \cong \det(P) \otimes_R \det(Q)$.*) So Picard group is a quotient of the Grothendieck group!*

3 Higher K-theory

Leads us to the question: how to define higher K-groups?

3.1 Whitehead group $K_1(R)$

Let R be an associative ring with unit. Include

$$\begin{aligned} \mathrm{GL}_n(R) &\to \mathrm{GL}_{n+1}(R) \\ g &\mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Let $\mathrm{GL}(R)$, take union of $\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_3(R) \hookrightarrow \cdots$.

Definition 3.1. $K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$ (abelianizing $\mathrm{GL}(R)$). By universal property of abelianization, any homomorphism $\mathrm{GL}(R) \to A$ (with A abelian) factors through $K_1(R)$.

Definition 3.2 (Steinberg module)**.** Let $n \geqslant 3$. The Steinberg module $\mathrm{St}_n(R)$ of a ring R. Group defined by generators $x_{ij}(r)$ with i, j a pair of distinct integers bw 1 and n, and $r \in R$ subject to Steinberg relations

(a) $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$.

(b) $[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & j \neq k, i \neq \ell \\ x_{i\ell}(rs) & j = k, i \neq \ell \text{ (smash)} \\ x_{kj}(-sr) & j \neq k, i = \ell \end{cases}$

Note that these relations are satisfied by elementary matrices $e_{ij}(r)$ in $\mathrm{GL}_n(R)$ (this matrix has a 1 in every diagonal spot, has an r in spot (i,j) $(i \neq j)$, 0 elsewhere).

Let $E_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by these elementary matrices. Turns out:

Prop 3.3. *For* $n \geqslant 3$ *and* R *commutative,* $E_n(R) \trianglelefteq \mathrm{GL}_n(R)$.

Lemma 3.4 (Whitehead's Lemma)**.** *$E(R)$ is the commutator subgroup of* $\mathrm{GL}(R)$. *So* $K_1(R) = \mathrm{GL}(R)/E(R)$.

Since Steinberg relations are satisfied by the elementary matrices, we have a canonical surjection

$$\phi_n : \mathrm{St}_n(R) \twoheadrightarrow E_n(R).$$

We have an injection $\mathrm{St}_n(R) \hookrightarrow \mathrm{St}_{n+1}(R)$, can write $\mathrm{St}(R) = \lim_{\rightarrow} \mathrm{St}_n(R) = \bigsqcup \mathrm{St}_n(R)/\sim$. By stabilizing ϕ_n, we get a surjection $\phi : \mathrm{St}(R) \to E(R)$. Define $K_2(R) = \ker \phi$. This yields an exact sequence of groups

$$1 \to K_2(R) \to \mathrm{St}(R) \xrightarrow{\phi} \mathrm{GL}(R) \to K_1(R) \to 1.$$

Turns out: $K_2(R) = Z(\mathrm{St}(R))$.

3.2 Topological tie-up

We've partially answered the question: We defined $K_1(R)$ and $K_2(R)$ algebraically. What does this have to do with topology? What does this have to do with K_0?

Definition 3.5 (Classifying space)**.** For a group G, construct a connected topological space BG whose $\pi_1 = G$ and higher homotopy groups vanish. ($H_*(G; M) \cong H_*(\mathrm{BG}; M)$ for M a G-module, homology with local coefficients).

Definition 3.6 (Quillen's + construction)**.** Take $G = \mathrm{GL}(R)$. Obtain the space $B\,\mathrm{GL}(R)$. Construct $B\,\mathrm{GL}(R)^+$, a CW complex X which has a distinguished map $B\,\mathrm{GL}(R) \to B\,\mathrm{GL}(R)^+$ such that

(a) $\pi_1(B\,\mathrm{GL}(R)^+) \cong K_1(R)$ (the abelianization of $\mathrm{GL}(R)$), and the natural map from $\mathrm{GL}(R) = \pi_1(B\,\mathrm{GL}(R))$ to $\pi_1(B\,\mathrm{GL}(R)^+)$ is surjective with kernel $E(R)$.

(b) $H_*(B\,\mathrm{GL}(R); M) \xrightarrow{\cong} H_*(B\,\mathrm{GL}(R)^+; M)$ for every $K_1(R)$-module M.

We can then define

$$K_n(R) := \pi_n(B\,\mathrm{GL}(R)^+).$$

This yields $K_1(R), K_2(R)$ as defined before! Would need to check it also gives $K_0(R)$!!