Configuration Spaces & Braid Groups

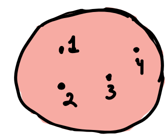
§1. Defining $\mathrm{Conf}_n(X)$, $\mathrm{UConf}_n(X)$

Def 1

Let X be a top. space. The (ordered) configuration space of n points on X is given by $\mathrm{Conf}_n(X) := \{(x_1,\dots,x_n) \in X^n \mid x_i \neq x_j \ \forall i \neq j\}$.
X^n without the "fat diagonal"

Topologize $\mathrm{Conf}_n(X)$ w/ the subspace topology on X^n.

We can think of this space as the collection of n distinct & labeled pts on X.



represents a single pt in config. space

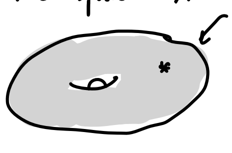
Forgetting a point yields a map $\mathrm{Conf}_{k+1}(X) \longrightarrow \mathrm{Conf}_k(X)$.

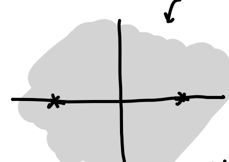
Def 2 If we forget the labels, we get the unordered config. space, $\mathrm{UConf}_n(X) := \mathrm{Conf}_n(X)/\Sigma_n$.
a mfld, b/c quotient of a mfld by a free action

Q: Why consider configuration spaces?

A: Provide good homeomorphism invariants of the underlying mfld! **!** config. spaces are NOT htpy invariants! **!**

Ex 1 (Knudsen notes) The spaces $X = T^2 \setminus \{pt\}$ & $Y = \mathbb{R}^2 \setminus S^0$





both deformation retract onto $S^1 \vee S^1$, so they're htpy equiv. But they're not homeomorphic! One can compute that $\mathrm{UConf}_2(X) \not\simeq \mathrm{UConf}_2(Y)$.

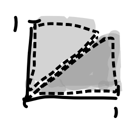
& if $X \cong Y$ then $\mathrm{Conf}_n(X) \cong \mathrm{Conf}_n(Y)$!

Ex 2 (Knudsen notes) Lens spaces. The cyclic group C_p acts on $S^3 \subseteq \mathbb{C}^2$ via $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1,\ e^{2\pi i q/p} z_2)$ for a chosen (p,q). Define $L(p,q) := S^3/C_p$ under this action.

A thm of Reidemeister $\Rightarrow$ $L(7,1) \simeq L(7,2)$ but they're not homeo. Their configuration spaces distinguish this!

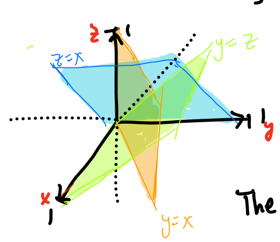
§1.1: Computing $\mathrm{Conf}_n(X)$ & $\mathrm{UConf}_n(X)$

Ex 1 $X = \mathbb{R} \overset{\text{homeo}}{\cong} (0,1)$. $\mathrm{Conf}_2((0,1)) = ((0,1))^2 \setminus \{(x,y) \mid x = y\}$.



two disjoint open 2-simplices.

What about $\mathrm{Conf}_3((0,1)) = ((0,1))^3 \setminus \{(x,y,z) \mid x \neq y,\ x \neq z,\ y \neq z\}$.



Given a coordinate $(x,y,z) \in \mathrm{Conf}_3((0,1))$, obtain an ordering based on the linear order of $x, y, z \in (0,1)$ (for instance, $y > x > z$).
$\updownarrow$ corresponds to a permutation
$(1\ 2) \in \Sigma_3$.

The assignment $\{\text{coordinate}\} \twoheadrightarrow \{\text{permutation}\}$ is locally constant by IVT.
(path-components are separated into $x>y>z$, $x>z>y$, $y>x>z$, $y>z>x$, $z>x>y$, $z>y>x$!)
$\Rightarrow \pi_0(\mathrm{Conf}_k((0,1))) \cong \Sigma_k$

What's $\mathrm{UConf}_n((0,1))$? Define a map $\mathrm{UConf}_n((0,1)) \longrightarrow \underset{\text{open } n\text{-simplex}}{\mathring{\Delta}^n} = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} \mid t_i > 0,\ \sum_{i=0}^n t_i = 1\}$

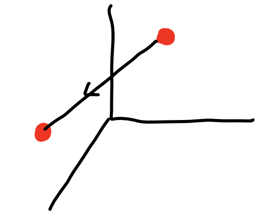
$\mathrm{UConf}_n((0,1)) = \mathrm{Conf}_n((0,1))/\Sigma_n$

$\{x_1,\dots,x_n\} \longmapsto (x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1}, 1 - x_n)$
ordered so that $x_1 < \dots < x_n$

This map is a homeomorphism (inverse map: $(t_0,\dots,t_n) \mapsto \{t_0, t_0+t_1, t_0+t_1+t_2, \dots, t_0+\dots+t_{n-1}\}$). So $\mathrm{UConf}_n((0,1)) \cong \mathring{\Delta}^n$, contractible b/c!

$\boxed{\text{Ex 2}}$ (K. 2.1.3) $X = \mathbb{R}^n$. What's $\mathrm{Conf}_2(\mathbb{R}^n)$?

Given $x_1 \neq x_2$, the configuration is determined by 3 pieces of data: the center of mass, the distance from each other, & the direction to each other.

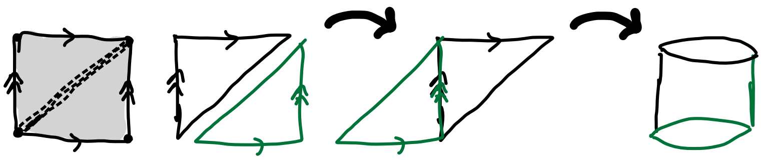


$$\begin{array}{ccc} \mathrm{Conf}_2(\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \times \mathbb{R}_{>0} \times S^{n-1} \\ (x_1, x_2) & \longmapsto & \left(\underset{\uparrow \atop \text{COM}}{\frac{x_1+x_2}{2}},\ \|x_2 - x_1\|,\ \frac{x_2-x_1}{\|x_2-x_1\|}\right) \end{array}$$

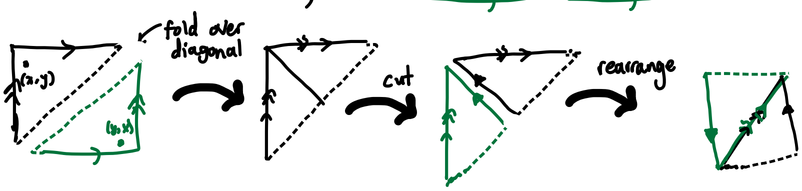
is a homeo, and the "Gauss map"

$\mathrm{Conf}_2(\mathbb{R}^n) \to S^{n-1}$ is a htpy equiv. Note that exchanging x_1 & x_2 on the LHS acts as the antipodal map on S^{n-1}, so $U\mathrm{Conf}_2(\mathbb{R}^n) = \mathrm{Conf}_2(\mathbb{R}^n)/\Sigma_2 \simeq \mathbb{R}P^{n-1}$.

$\boxed{\text{Ex 3}}$ (K. 2.1.4). $\mathrm{Conf}_2(S^1) = \{(x,y) \in S^1 \times S^1 \mid x \neq y\}$
$\cong S^1 \times I$



What's $U\mathrm{Conf}_2(S^1) = \mathrm{Conf}_2(S^1)/\Sigma_2$?



= open mobius strip

§1.2 - Forgetting a Point

Let M be a mfld. Natural projections from the product factor through the config. space

$$\begin{tikzcd} \mathrm{Conf}_\ell(M) \arrow[r, hook] \arrow[d, dashed] & M^\ell \arrow[d] & (x_1, \dots, x_\ell) \arrow[d, mapsto] \\ \mathrm{Conf}_k(M) \arrow[r, hook] & M^k & (x_1, \dots, x_k) \end{tikzcd}$$

fiber is $\mathrm{Conf}_{\ell-k}(M \setminus \{x_1, \dots, x_k\})$

$\boxed{\text{Claim}}$ The "forgetting a point" map $\mathrm{Conf}_{k+1}(M) \longrightarrow \mathrm{Conf}_k(M)$ is a fiber bundle.
$(x_1, \dots, x_{k+1}) \longmapsto (x_1, \dots, x_k)$

$\boxed{\text{Fadell-Neuwirth Thm 1}}$ $\pi: \mathrm{Conf}_n(M) \longrightarrow M$ ($n>1$) is a locally trivial fiber bundle, with fiber $\mathrm{Conf}_{n-1}(M \setminus \{\underset{\text{some pt in } M}{y}\})$.

Pf: ① π is locally trivial: Let $y \in M$ be a chosen pt, and fix some other pt $x_0 \in M$. Let $\alpha: M \to M$ be a homeo s.t. $\alpha(y) = x_0$. Let U be a Euclidean nbhd of x_0. Let $\theta: U \times \bar{U} \to \bar{U}$ denote a map w/ the following properties:

i.) $\forall x \in U$, $\theta(x, -): \bar{U} \to \bar{U}$ is a homeo leaving $\partial\bar{U}$ fixed.

ii.) $\forall x \in U$, $\theta(x,x) = x_0$.

Then we can extend θ to $U \times M \to M$ by setting $\theta(x,y) = y$ $\forall y \notin U$. Obtain the local product representation via

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, shift left, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathrm{Conf}_{n-1}(M \setminus ?) \arrow[ll, shift left, "\varphi^{-1}"] \arrow[dl, "\text{proj.}"] \\ & U & \end{tikzcd}$$

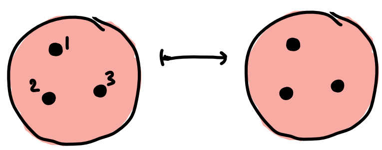
$\varphi(x, p_2, \dots, p_n) = (x, \theta(x,-)^{-1}(\alpha(p_2)), \dots, \theta(x,-)^{-1}(\alpha(p_n)))$

$\varphi^{-1}(x, p_2, \dots, p_n) = (x, \alpha^{-1}(\theta(x, p_2)), \dots, \alpha^{-1}(\theta(x, p_n)))$.

Can extend this proof to get $\boxed{\text{Thm 3}}$ $\pi: \mathrm{Conf}_n(M) \longrightarrow \mathrm{Conf}_{n-1}(M)$ is a locally trivial fiber bundle w/ fiber
$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1})$
$\mathrm{Conf}_1(M \setminus \{x_1, \dots, x_{n-1}\})$.

§1.3 - Forgetting the ordering.

Saw already that we have a map $\mathrm{Conf}_n(M) \longrightarrow U\mathrm{Conf}_n(M)/\Sigma_n$. This is also a fiber bundle! (w/ discrete fibers, so a covering space).



$\boxed{\text{Cor (K. 2.3.4)}}$ Suppose M is a simply-connected n-mfld for $n \geq 3$. Then $Conf_k(M)$ is simply-connected $\forall k \geq 0$. In particular, $\pi_1(UConf_k(M)) \cong \Sigma_k$.

Pf: By Van Kampen's & the assumption on n, the space $M \setminus \{pt\}$ is s.c. The fiber bundle $M\setminus\{pt\} \to Conf_k(M) \to Conf_{k-1}(M)$ and induction (noting that cases $k=0,1$ are trivial), LES of htpy grps $\pi_1(M\setminus\{pt\}) \to \pi_1(Conf_k(M)) \to \pi_1(Conf_{k-1}(M))$
$\Rightarrow \uparrow = 0$.

Second part follows from $Conf_k(M) \to UConf_k(M)$ being a universal cover w/ deck group $= \Sigma_k$.

§1.4 – $K(\pi,1)$'s

$\boxed{\text{Cor}}$ (K. 2.3.5)

Let M be a connected surface other than S^2 or $\mathbb{RP}^2$. Then $Conf_k(M)$ is aspherical for every $k \geq 0$. In particular, $UConf_k(M)$ is aspherical.

higher htpy groups vanish

Pf: The case $k=0$ is trivial. The case $k=1$ is by assumption on M, since connected surfaces other than S^2 & $\mathbb{RP}^2$ are aspherical (orientable compact surfaces of genus ≥ 1 are covered by $\mathbb{R}^2$, nonorientable surfaces of genus ≥ 2 are covered by $\mathbb{R}^2$)
or $\mathbb{H}^2$ "Poincaré polygon thm" how many $\mathbb{RP}^2$s in the connect sum

Also the assumption on M is s.t. $M \setminus \{pt\}$ is aspherical (why?) **?**
Wedge of circles?

So induction + exact sequence $\pi_i(M\setminus pt) \to \pi_i(Conf_k(M)) \to \pi_i(Conf_{k-1}(M))$ for $i \geq 2$ $\Rightarrow$ $Conf_k(M)$ is aspherical $\forall k \geq 0$.

Now $Conf_k(M) \to UConf_k(M)$ is a covering space $\Rightarrow$ $\pi_i(UConf_k(M)) = 0$ for $i \geq 2$. $\square$

In particular, this implies that $Conf_k(M)$ & $UConf_k(M)$ are $K(\pi,1)$'s. $\boxed{\text{Q: What is } \pi\text{?}}$

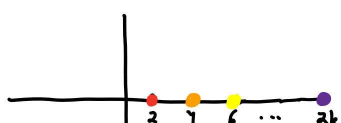
§2 - Braid Groups

$\boxed{\text{Def 3}}$ The braid groups $B_n(M)$ of a mfld M are the groups $B_n(M) := \pi_1(UConf_n(M))$.
unordered!

The pure braid groups $PB_n(M)$ are the grps $PB_n(M) := \pi_1(Conf_n(M))$.
ordered!

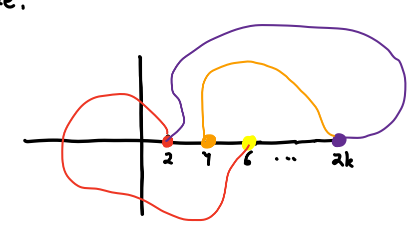
§2.1 – $M = \mathbb{R}^2$ & Artin's Braid Group

Q: How do we compute $\pi_1(UConf_k(\mathbb{R}^2))$?
Fix a basepoint $\{(2i,0)\}_{i=1}^k$. (*)



An elt of $\pi_1(UConf_k(\mathbb{R}^2))$ is a path in $UConf_k(\mathbb{R}^2)$ starting and ending at our basepoint (*).
But the order of the colors can change since we're in unordered configuration space.
This means: an element of $\pi_1(UConf_k(\mathbb{R}^2))$ is determined by the data of:

(i) a permutation $\tau \in \Sigma_k$
(ii) a path $p: [0,\pi] \to (\mathbb{R}^2)^k$ s.t.
(1) $(p(0))_r = (2r, 0)$ $\forall\ 1 \leq r \leq k$
(2) $(p(1))_r = (2\cdot\tau(r), 0)$ $\forall\ 1 \leq r \leq k$
(3) $(p(t))_r \neq (p(t))_{r'}$ $\forall\ 1 \leq r \neq r' \leq k$ & $t \in [0,\pi]$.



Obtain a canonical group homomorphism $\pi_1(UConf_k(\mathbb{R}^2)) \to \Sigma_k$ which we can show is surjective. The kernel consists of those paths which start & end at the same basepoint on the nose, i.e., $\pi_1(Conf_k(\mathbb{R}^2))$.
(clear that $\pi_1(Conf_k(\mathbb{R}^2)) \leq \ker$, & the other containment follows from an index argument & that $Conf_k(\mathbb{R}^2) \to UConf_k(\mathbb{R}^2)$ is a $k!$-sheeted cover.)

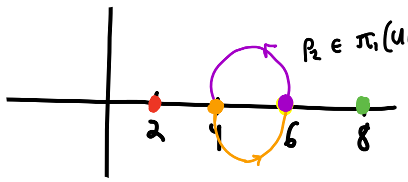
So we have a SES of grps $1 \to PB_k \to B_k \to \Sigma_k \to 1$.

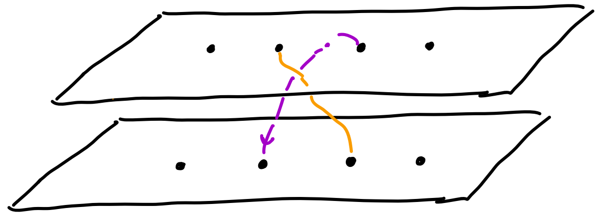
$\boxed{\text{Claim}}$ $B_k \twoheadrightarrow \Sigma_k$. (K. 2.4.2)

~~Pf~~: Since this map is a grp hom., STS generators can be attained. NTS $\exists \rho_i \in B_k$ lifting $(i\ i{+}1)$.

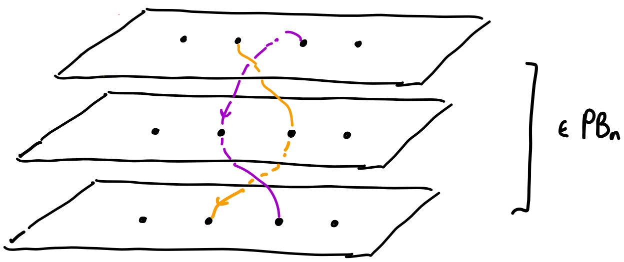
For $1 \le i \le r$, define a path $\rho_i : [0,\pi] \to (\mathbb{R}^2)^k$ via

$$(\rho_i(t))_r = \begin{cases} (2r, 0) & r \notin \{i, i{+}1\} \\ (2i{+}1 + \cos(t+\pi),\ \sin(t+\pi)) & r = i \\ (2i{+}1 + \cos(t),\ \sin(t)) & r = i{+}1 . \end{cases}$$





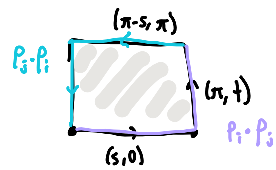
stack $\mathbb{R}^2$'s
imagine strands
moving through



visual demonstration
that loops in $Conf_n(\mathbb{R}^2)$ &
$UConf_n(\mathbb{R}^2)$ correspond to braids

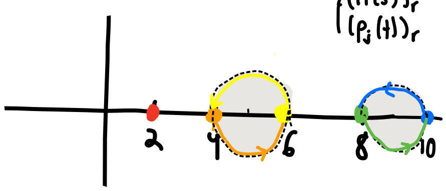
Turns out: if $|i-j|>1$, then ρ_i & ρ_j commute.

$\boxed{\text{Lemma}}$ (K. 2.4.3) If $|i-j|>1$, $\rho_i \rho_j = \rho_j \rho_i$.



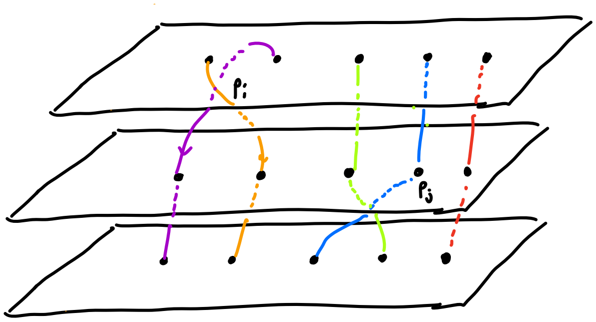
~~Pf~~: WLOG $j > i$.

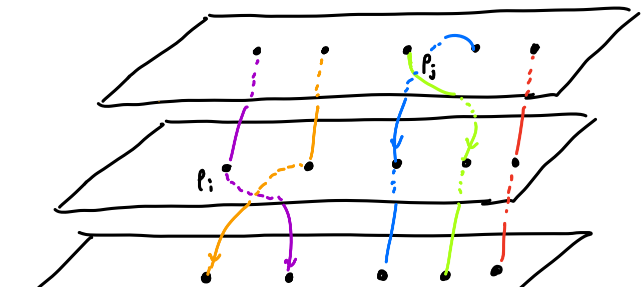
$$H(s,t)_r = \begin{cases} (2r,0) & r \notin \{i, i{+}1, j, j{+}1\} \\ (\rho_i(s))_r & r \in \{i, i{+}1\} \\ (\rho_j(t))_r & r \in \{j, j{+}1\} \end{cases}$$



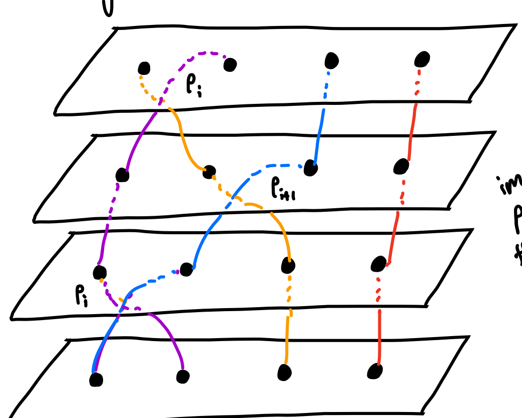
So $H(s,0)_r = \begin{cases} (2r,0) & r \notin \{i,i{+}1\} \\ \rho_i(s)_r & r = i, i{+}1 \end{cases}$ $\quad H(\pi,t)_r = \begin{cases} (2r,0) & r \notin \{i,i{+}1,j,j{+}1\} \\ \tau_i(r) & r = i, i{+}1 \\ \rho_j(t)_r & r = j, j{+}1 \end{cases}$

H defines a htpy b/w $\rho_i \cdot \rho_j$ & $\rho_j \cdot \rho_i$!

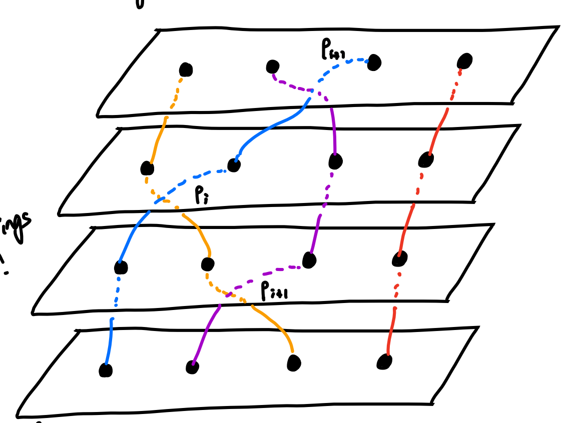




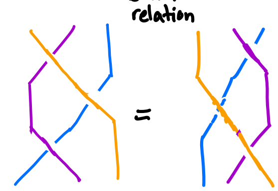
Maybe the above is believable? What about braiding adjacent strands?



imagine pulling the strings taut!



braid relation



\ goes over / goes under

(! for inverses - reflect across $\mathbb{R}^2 \times \{1\}$)

$\boxed{\text{Lemma}}$ (K. 24.4) $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \quad \forall 1 \le i \le k$.
~~Pf~~: omitted.

group of "isotopy classes" of braids

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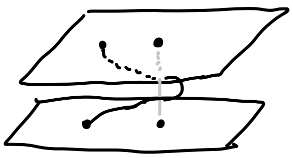
What does this give us? Artin's braid group! $B_n = \langle \rho_1, \dots, \rho_{n-1} \mid \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1},\ \rho_i \rho_j = \rho_j \rho_i \text{ for } |i-j|>1 \rangle$

(Can do a similar thing for pure braid groups. this time generators are $T_{ij} = T_{ji}$ where you wrap string i around string j counterclockwise:



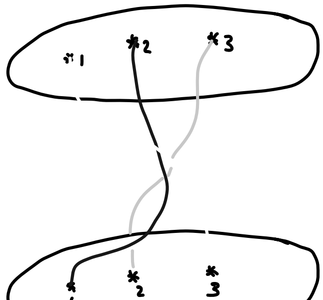
There are finitely many relations and they are all central!

Recall the mapping class group.

$\boxed{\text{Def}}$ Let M be a (smooth) oriented manifold. Let $\mathrm{Diff}^+(M, \partial M) = \{$ orientation-preserving diffeos $M \to M$ fixing ∂M ptwise $\}$. Topologize $\mathrm{Diff}^+(M)$ w/ the compact-open topology. Then path-components in $\mathrm{Diff}^+(M) \longleftrightarrow$ isotopy classes of diffeos (i.e., $f \sim g$ are isotopic if $\exists$ htpy of f & g which is a diffeo & fixes ∂M at every time t). Define $\mathrm{MCG}(M) := \mathrm{Diff}^+(M)/\sim$ $\underset{\mathrm{Mod}(M)}{\shortparallel}$
(equiv., $\mathrm{MCG}(M) = \pi_0(\mathrm{Diff}^+(M))$.

$\boxed{\text{Fact}}$ $\mathrm{MCG}(D^2) = \{e\}$ (triv.)

Let $D_n = D^2$ with n marked points.

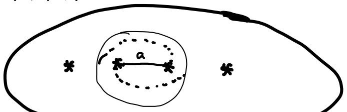


Let Φ be a homeo of $(D^2, \partial D^2)$ that leaves invariant the n marked pts. This is a homeo of $(D^2, \partial D^2)$, so by $\boxed{\text{fact}}$ we know $\Phi \sim \mathrm{id}$. During this isotopy the marked pts move around the interior & then return to where they started.

Gives $B_n \cong \mathrm{Mod}(D_n)$

Under this isomorphism, where are generators ρ_i mapped?

$\rho_i \longmapsto$ half twist H_α



(see Ch. 9 of Farb-Margalit "A Primer")

§3 – Integral Burau Representation

$\boxed{\text{Def}}$ The unreduced Burau rep. can be given explicitly by
$$B_n \longrightarrow GL_n(\mathbb{Z}[t, t^{-1}])$$
$$\sigma_i \longmapsto \begin{pmatrix} I_{i-1} & & \\ & \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} & \\ & & I_{n-1-i} \end{pmatrix}.$$

The integral Burau rep. is the specialization to $t = -1$. (The more general map comes from the action of B_n on $H_1(D_n) \dots$)

Consider now the composition $B_n \xrightarrow{\rho} GL_n(\mathbb{Z}) \xrightarrow{\text{mod } m} GL_n(\mathbb{Z}/m)$. Let $B_n[m] := \ker(\text{comp.})$

level m congruence subgroup.

$\boxed{\text{Thm (Arnol'd) '68}}$ $B_n[2] = PB_n$!

$\boxed{\text{Thm (Brendle-Margalit) '14}}$ $B_n[4] = PB_n^2$ (squares of elts in PB_n!)

$\boxed{\text{Thm (Brendle-Margalit) '14}}$ $PB_{2g+1} \twoheadrightarrow Sp_{2g}(\mathbb{Z})[2]$ guys which are the identity mod 2.